DISQUISITIONES
DE
CURVATURIS NONNULLARUM SECUNDI ORDINIS SUPERFICIERUM.

DISSERTATIO
INAUGURALIS MATHEMATICA

QUAM
CONSENSU ET AUCTORITATE
AMPLISSIMI
PHILOSOPHORUM ORDINIS
IN
ALMA LITERARUM UNIVERSITATE
GRYPHICA
AD SUMMOS
IN PHILOSOPHIA HONORES
RITE IMPETRANDOS
CALENDIS MARTIIS ANNI MDCCCLXII
UNA CUM THESIBUS ADIECTIS
PUBLICE DEFENDET
AUCTOR

AXELIUS THEODORUS WILLERT,
POMERANUS.

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Generales de curvaturis superficierum curvarum disquisitiones multi iique subtilissimi mathematici instituerunt multaque theoremata invenerunt memorabilia. Ad speciales autem superficies curvas, quantum equidem scio, theoremata illa raro adhibita sunt, id quod eo magis mirum videtur, quod multis curvis superficiebus tam memorabiles formulae atque elegantes efficiuntur. Ratio quidem, qua paene usque ad hanc diem disquisitiones generales haud paucis permutationibus coordinatarum adhibitis factae sunt, re vera minus apta fuit, quae ad speciales superficies curvas traduceretur; qua de re praecipue theoremata per generalem disquisitionem inventa in speciales curvas superficies raro translata mihi videntur esse. Quod quum v. ill. Grunertus cognovisset atque perspe-

xisset, rem aliter aggressus ea tractavit ratione, quae ad speciales superficies curvas summa facilitate possit applicari, quod speramus fore ut hoc opusculo manifestum fiat. Ab iis enim, quas v. ill. **Grunertus** anno 57. h. s. in diario suo*) publicavit, formulis profecti de curvaturis ellipsoidis, hyperboloidum, quae à une nappe et à deux nappes nominantur, paraboloidum et ellipticae et hyperbolicae disquisitiones instituimus. Disquisitionem **Grunertianam** illam prorsus notam ponemus.

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*) Vid. Archiv der Mathematik und Physik. T. XXVIII. pag. 163 sqq.

Ellipsoides.

I.

Ellipsoidis cujuslibet aequationem si ponamus generalem

$$u = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 - 1 = 0, \text{*)}$$

functione u partialiter ad x, y, z differentiata primae et secundae partiales quotientes differentiales functionis u efficiuntur:

$$\frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = \frac{2y}{b^2}, \quad \frac{du}{dz} = \frac{2z}{c^2};$$

$$\frac{d^2u}{dx^2} = \frac{2}{a^2}, \quad \frac{d^2u}{dy^2} = \frac{2}{b^2}, \quad \frac{d^2u}{dz^2} = \frac{2}{c^2};$$

$$\frac{d^2u}{dxdy} = 0, \quad \frac{d^2u}{dydz} = 0, \quad \frac{d^2u}{dzdx} = 0.$$

Coordinatae puncti cujusvis in superficie ellipsoidis siti non, ut in theoria generali, per (xyz) sed per (uvw), coordinatae vero variabiles non, ut ibidem factum est, per (XYZ), sed per (xyz) designentur. Quare in deducendis ellipsoidis formulis ex generalibus pro (xyz) (uvw), pro (XYZ) (xyz), pro quotientibus vero differentialibus $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$ valores $\frac{2u}{a^2}$, $\frac{2v}{b^2}$, $\frac{2w}{c^2}$ ponendi sunt.

Ex generali igitur theoria aequatio plani ellipsoidem in puncto (uvw) tangentis est:

$$\frac{u}{a^2}(x-u) + \frac{v}{b^2}(y-v) + \frac{w}{c^2}(z-w) = 0,$$

vel, quum sit

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2 = 1,$$

$$\frac{ux}{a^2} + \frac{vy}{b^2} + \frac{wz}{c^2} = 1.$$

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*) De ellipsoide cfr. Grunerti geom. analyt. part. II. §. 77. p. 212.

Per punctum (uvw) ellipsoidis planum aequatione

$$A(x-u)+B(y-v)+C(z-w)=0$$

definitivum ponemus et curvam, quae plano ellipsoidem persecante efficitur, simpliciter sectionem nominabimus. Qua sectione abhinc occupabimur.

Ex theoria universali aequationes lineae, quae sectionem in puncto (uvw) tangit, facile invenimus has:

$$\frac{x-u}{B\frac{w}{c^2}-C\frac{v}{b^2}}=\frac{y-v}{C\frac{u}{a^2}-A\frac{w}{c^2}}=\frac{z-w}{A\frac{v}{b^2}-B\frac{u}{a^2}};$$

hanc plani normalis:

$$\{B\frac{w}{c^2}-C\frac{v}{b^2}\}(x-u)+\{C\frac{u}{a^2}-A\frac{w}{c^2}\}(y-v)+\{A\frac{v}{b^2}-B\frac{u}{a^2}\}(z-w)=0;$$

has normalis in eodem sectionis puncto erectae:

$$\frac{x-u}{A(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{u}{a^2}}$$

$$=\frac{y-v}{B(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{v}{b^2}}$$

$$=\frac{z-w}{C(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{w}{c^2}}.$$

Deinde quum quantitates brevitatis causa in theoriam generalem inductae ita exsistant:

$$\tfrac{1}{2}U=A(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{u}{a^2},$$

$$\tfrac{1}{2}V=B(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{v}{b^2},$$

$$\tfrac{1}{2}W=C(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2})-(A^2+B^2+C^2)\frac{w}{c^2},$$

$$\tfrac{1}{4}Q^2=(A^2+B^2+C^2)\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\}-\{A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\}^2$$

$$=\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2+\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2,$$

$$\tfrac{1}{8}\Theta=\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2+\frac{1}{c^2}\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2,$$

hae obtinentur aequationes, quibus coordinatae centri (XYZ) et radius (R) curvaturae in puncto (uvw) sectionis sunt definita:

$$\frac{X-u}{A\left(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\right)-(A^2+B^2+C^2)\frac{u}{a^2}}$$

$$=\frac{Y-v}{B\left(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\right)-(A^2+B^2+C^2)\frac{v}{b^2}}$$

$$=\frac{Z-w}{C\left(A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\right)-(A^2+B^2+C^2)\frac{w}{c^2}}$$

$$=\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}-\left\{A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\right\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\left(B\frac{w}{c^2}-C\frac{v}{b^2}\right)^2+\frac{1}{b^2}\left(C\frac{u}{a^2}-A\frac{w}{c^2}\right)^2+\frac{1}{c^2}\left(A\frac{v}{b^2}-B\frac{u}{a^2}\right)^2\right]};$$

$$R=$$

$$\frac{\left[(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}-\left\{A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2+\frac{1}{b^2}\left\{C\frac{u}{a^2}-A\frac{w}{c^2}\right\}^2+\frac{1}{c^2}\left\{A\frac{v}{b^2}-B\frac{u}{a^2}\right\}^2\right]}.$$

Ratione habita aequationis

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2+\left(\frac{w}{c}\right)^2=1$$

facili calculo expressionem, quam pro $(\frac{1}{8}\Theta)$ dedimus, transmutamus in hanc

$$\frac{A^2a^2+B^2b^2+C^2c^2-\{Au+Bv+Cw\}^2}{a^2b^2c^2},$$

quae in formulis superioribus substitui potest, si quando usui est.

II.

Jam vero sectiones normales considerabimus.

Si supponimus, planum aequatione

$$A(x-u)+B(y-v)+C(z-w)=0$$

definitum per normalem in puncto (uvw) ellipsoidis erectam transire et igitur sectionem describere normalem, ex analyticae geometriae principiis debet esse

$$A\frac{u}{a^2}+B\frac{v}{b^2}+C\frac{w}{c^2}=0.$$

Itaque sunt in hoc casu:

$$\tfrac{1}{2}U=-\{A^2+B^2+C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V=-\{A^2+B^2+C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W=-\{A^2+B^2+C^2\}\frac{w}{c^2},$$

$$\tfrac{1}{4}Q^2=(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}$$

et ad coordinatas centri (XYZ) et radium (R) curvaturae sectionis normalis definienda obtinemus aequationes:

$$X-u=-\frac{u}{a^2}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2+\frac{1}{c^2}\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2},$$

$$Y-v=-\frac{v}{b^2}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2+\frac{1}{c^2}\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2},$$

$$Z-w=-\frac{w}{c^2}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2+\frac{1}{c^2}\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2};$$

$$R=\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}{\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}-A\frac{w}{c^2}\}^2+\frac{1}{c^2}\{A\frac{v}{b^2}-B\frac{u}{a^2}\}^2};$$

ex quibus statim consequuntur:

$$X-u=-\frac{u}{a^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Y-v=-\frac{v}{b^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Z - w = -\frac{w}{c^2} \cdot \frac{R}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}};$$

ita ut hoc in casu etiam sit:

$$\frac{R\frac{u}{a^2}}{X-u} = \frac{R\frac{v}{b^2}}{Y-v} = \frac{R\frac{w}{c^2}}{Z-w} = -\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}.$$

Denotante littera P perpendiculum ab initio coordinatarum in plano, quod ellipsoidem in puncto (uvw) tangit, demissum, ex geometria analytica est

$$\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}} = \frac{1}{P}.$$

Quare perpendiculo substituto aequationes datae has praebent formas:

$$X - u = -\frac{u}{a^2} \cdot PR,$$

$$Y - v = -\frac{v}{b^2} \cdot PR,$$

$$Z - w = -\frac{w}{c^2} \cdot PR$$

et

$$\frac{R\frac{u}{a^2}}{X-u} = \frac{R\frac{v}{b^2}}{Y-v} = \frac{R\frac{w}{c^2}}{Z-w} = -\frac{1}{P}.$$

In plano ellipsoidem in puncto (uvw) tangente ducatur per punctum (uvw) linea recta, cujus aequationes sunt

$$\frac{x-u}{\cos\theta} = \frac{y-v}{\cos\omega} = \frac{z-w}{\cos\varpi},$$

ubi per (θ, ω, ϖ) designantur anguli 180^0 non superantes, qui linea ea et positivis axium coordinatarum partibus efficiuntur.

Qua linea et normali in puncto (uvw) ellipsoidis erecta situm plani sectionis definitum esse si ponimus, ex theoria generali ad coordinatas centri (XYZ) et radium (R) curvaturae sectionis definienda has obtinemus aequationes:

$$X - u = -\frac{\frac{u}{a^2}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\varpi^2}{c^2}},$$

$$Y - v = -\frac{\frac{v}{b^2}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\overline{\omega}^2}{c^2}},$$

$$Z - w = -\frac{\frac{w}{c^2}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\overline{\omega}^2}{c^2}};$$

$$R = \frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\overline{\omega}^2}{c^2}}.$$

Alteri sectioni normali per punctum (uvw) ellipsoidis positae, quam eodem modo, ut praecedentem, sed angulis $(\theta_1, \omega_1, \overline{\omega}_1)$ determinatam esse statuimus, hae prodeunt formulae, in quibus significationem symbolorum nulla definitione data satis perspicuam fore puto,

$$X_1 - u = -\frac{\frac{u}{a^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Z_1 - w = -\frac{\frac{w}{c^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2}};$$

$$R_1 = \frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2}}.$$

Quas duas sectiones normales si inter se ad perpendiculum directas esse et brevitatis causa

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\overline{\omega}^2}{c^2} = \tfrac{1}{2}\Omega,$$

$$\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2} = \tfrac{1}{2}\Omega_1,$$

statuimus, theoria docet generalis hanc inter angulos $(\theta, \omega, \overline{\omega})$ et $(\theta_1, \omega_1, \overline{\omega}_1)$ interesse aequationem

$$\tfrac{1}{2}\Omega_1 = \frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}},$$

unde, quum sit (vid. Grunerti Arch. XXVIII. p. 178.)

$$\left(\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right)\left\{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\overline{\omega}^2}{c^2}\right\} - \left\{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}\right\}$$

$$= \frac{1}{a^2}\left\{\frac{v}{b^2}\cos\overline{\omega} - \frac{w}{c^2}\cos\omega\right\}^2 + \frac{1}{b^2}\left\{\frac{w}{c^2}\cos\theta - \frac{u}{a^2}\cos\overline{\omega}\right\}^2$$

$$+ \frac{1}{c^2}\left\{\frac{u}{a^2}\cos\omega - \frac{v}{b^2}\cos\theta\right\}^2;$$

ratione habita aequationis

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2 = 1$$

facili calculo expressionem invenimus memorabilem:

$$\tfrac{1}{2}\Omega_1 = \frac{a^2\cos\theta^2 + b^2\cos\omega^2 + c^2\cos\overline{\omega}^2 - \{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}^2}{a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}.$$

Quare in hoc casu ad quantitates $(X_1, Y_1, Z_1; R_1)$ definiendas obtinemus aequationes, quibus quantitates illae ab angulis $(\theta, \omega, \overline{\omega})$ dependentes datae sunt:

$$X_1 - u = -\frac{\frac{u}{a^2}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= -\frac{ub^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{a^2\cos\theta^2 + b^2\cos\omega^2 + c^2\cos\overline{\omega}^2 - \{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}^2},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= -\frac{vc^2 a^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}}{a^2 \cos\theta^2 + b^2 \cos\omega^2 + c^2 \cos\varpi^2 - \{u\cos\theta + v\cos\omega + w\cos\varpi\}^2},$$

$$Z_1 - w = -\frac{\frac{w}{c^2}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\varpi^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= -\frac{wa^2 b^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}}{a^2 \cos\theta^2 + b^2 \cos\omega^2 + c^2 \cos\varpi^2 - \{u\cos\theta + v\cos\omega + w\cos\varpi\}^2};$$

$$R_1 = \frac{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{1}{2}}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\varpi^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= \frac{a^2 b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{3}{2}}}{a^2 \cos\theta^2 + b^2 \cos\omega^2 + c^2 \cos\varpi^2 - \{u\cos\theta + v\cos\omega + w\cos\varpi\}^2}.$$

Ex formulis superioribus statim sequuntur:

$$\frac{1}{R} = \frac{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\varpi^2}{c^2}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \frac{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} + \frac{\sin\varpi^2}{c^2}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{1}{2}}} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{3}{2}}}$$

Itaque est:

$$\frac{1}{R} + \frac{1}{R_1} = \frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{1}{2}}} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} + \frac{w^2}{c^6}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{3}{2}}}$$

$$= \frac{\frac{u^2}{a^4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{v^2}{b^4}\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + \frac{w^2}{c^4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}^{\frac{3}{2}}}$$

$$=\frac{\left(\frac{u}{a}\right)^2(b^2+c^2)+\left(\frac{v}{b}\right)^2(c^2+a^2)+\left(\frac{w}{c}\right)^2(a^2+b^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

unde ratione habita aequationis

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2+\left(\frac{w}{c}\right)^2=1$$

statim obtinemus expressionem memorabilem atque elegantem:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{a^2+b^2+c^2-(u^2+v^2+w^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

ex qua summam radiorum curvaturarum inversorum, qui ad binas per idem ellipsoidis punctum positas et ad perpendiculum inter se directas sectiones normales relati sunt, in omni ellipsoidis puncto constantem esse apparet.

Porro est

$$\frac{1}{R}-\frac{1}{R_1}=\frac{\frac{\cos 2\theta}{a^2}+\frac{\cos 2\omega}{b^2}+\frac{\cos 2\overline{\omega}}{c^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}+\frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

Quia ex aequationibus supra datis sunt:

$$\frac{1}{X-u}+\frac{1}{X_1-u}=-\frac{a^2}{u}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\},$$

$$\frac{1}{Y-v}+\frac{1}{Y_1-v}=-\frac{b^2}{v}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\},$$

$$\frac{1}{Z-w}+\frac{1}{Z_1-w}=-\frac{c^2}{w}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\};$$

in omni ellipsoidis puncto binis sectionibus normalibus, quae inter se directae sunt, summae quoque

$$\frac{1}{X-u}+\frac{1}{X_1-u};\quad \frac{1}{Y-v}+\frac{1}{Y_1-v};\quad \frac{1}{Z-w}+\frac{1}{Z_1-w}$$

constantes prodeunt.

III.

Sectionibus normalibus, quae ad curvaturam maximum et minimum praebeant, abhinc occupabimur. Quas sectiones brevitatis causa principales vocabimus.

Si supponimus, sectiones duas normales, de quibus quaesivimus, principales esse ellipsoidis, ex theoria generali aequationes obtinemus:

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}+\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}$$

vel, quod jam supra demonstravimus,

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{a^2+b^2+c^2-(u^2+v^2+w^2)}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}$$

et

$$\tfrac{1}{4}\Omega\,\Omega_1=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-1};$$

ex quibus facili calculo invenimus:

$$\Omega=\frac{\left\{\begin{array}{l}a^2+b^2+c^2-(u^2+v^2+w^2)\\ \pm\sqrt{\{a^2+b^2+c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}},$$

$$\Omega_1=\frac{\left\{\begin{array}{l}a^2+b^2+c^2-(u^2+v^2+w^2)\\ \mp\sqrt{\{a^2+b^2+c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}},$$

ubi signa superiora et inferiora in se invicem referenda sunt.

Ex aequationibus

$$R=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}+\frac{\cos\overline{\omega}^2}{c^2}}$$

et

$$\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}+\frac{\cos\overline{\omega}^2}{c^2}=\tfrac{1}{2}\Omega$$

sequitur:

$$\frac{1}{R}=\frac{\tfrac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}.$$

Eodem modo invenitur:

$$\frac{1}{R_1}=\frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}.$$

Sunt igitur:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{\frac{1}{2}(\Omega+\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}$$

et

$$\frac{1}{R\,R_1}=\frac{\frac{1}{4}\Omega\Omega_1}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}};$$

vel, si loco Ω et Ω_1 valores antea inventi substituuntur,

$$\frac{1}{R}+\frac{1}{R_1}=\frac{a^2+b^2+c^2-(u^2+v^2+w^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}$$

et

$$\frac{1}{R\,R_1}=\frac{1}{a^2\,b^2\,c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-2};$$

vel, si P tantum, quantum antea, hic quoque valet,

$$\frac{1}{R}+\frac{1}{R_1}=P^3.\frac{a^2+b^2+c^2-(u^2+v^2+w^2)}{a^2\,b^2\,c^2}$$

et

$$\frac{1}{R\,R_1}=\frac{P^4}{a^2\,b^2\,c^2}$$

vel

$$R\,R_1=\frac{a^2\,b^2\,c^2}{P^4}$$

vel

$$a^2\,b^2\,c^2=R\,R_1\,P^4$$

vel etiam

$$P=\sqrt[4]{\frac{a^2\,b^2\,c^2}{R\,R_1}},$$

quam expressionem memorabilem, quantum equidem scio, jam Dupin invenit.

Est porro

$$R+R_1=\left\{\frac{1}{R}+\frac{1}{R_1}\right\}R\,R_1,$$

ergo, si loco $\left(\frac{1}{R}+\frac{1}{R_1}\right)$ et $R R_1$ valores datos substituimus,

$$R+R_1=\{a^2+b^2+c^2-(u^2+v^2+w^2)\}\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}$$

vel

$$R+R_1=\frac{a^2+b^2+c^2-(u^2+v^2+w^2)}{P}.$$

Ex aequatione

$$R R_1=a^2 b^2 c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^2,$$

quam si per aequationem

$$R=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}+\frac{\cos\overline{\omega}^2}{c^2}}$$

dividimus, pro radio R_1 expressionem

$$R_1=a^2 b^2 c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}\cdot\left\{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}+\frac{\cos\overline{\omega}^2}{c^2}\right\}$$

effici, mihi videtur dignum esse, quod commemorem.

Denique si in aequationibus

$$\frac{1}{R}=\frac{\frac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}$$

et

$$\frac{1}{R_1}=\frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}$$

loco $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ valores eorum substituimus, pro radiis sectionum principalium osculatoriis memorabiles obtinemus expressiones, in quibus praeter quantitates constantes (abc) coordinatae modo (uvw) reperiuntur et signa superiora ad superiora, inferiora ad inferiora referenda sunt,

$$\frac{2}{R}=\frac{\left\{\begin{array}{l}a^2+b^2+c^2-(u^2+v^2+w^2)\\ \pm\sqrt{\{a^2+b^2+c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

$$\frac{2}{R_1}=\frac{\left\{\begin{array}{l}a^2+b^2+c^2-(u^2+v^2+w^2)\\ \mp\sqrt{\{a^2+b^2+c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

Sectiones principales ad Ω et Ω_1 quantitates, alteram maximum, alteram minimum praebere, theoria universali novimus. Conditiones, quibus Ω et Ω_1 maxima et minima sunt, nunc constituemus.

Ex theoria generali sunt:

$$\frac{d^2\Omega}{d\,\theta^2}=2.\sin\theta^2.\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}-\frac{w}{c^2}\cos\omega\right\}^2}(\Omega_1-\Omega),$$

$$\frac{d^2\Omega_1}{d\theta_1{}^2}=2.\sin\theta_1{}^2.\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}_1-\frac{w}{c^2}\cos\omega_1\right\}^2}(\Omega-\Omega_1),$$

vel, quum facillime inveniatur

$$\Omega-\Omega_1=2.\frac{R_1-R}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

substituendo obtinemus:

$$\frac{d^2\Omega}{d\,\theta^2}=4\sin\theta^2\frac{R-R_1}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}.\left\{\frac{v}{b^2}\cos\overline{\omega}-\frac{w}{c^2}\cos\omega\right\}^2},$$

$$\frac{d^2\Omega_1}{d\,\theta_1{}^2}=4.\sin\theta_1{}^2\frac{R_1-R}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}\left\{\frac{v}{b^2}\cos\overline{\omega}_1-\frac{w}{c^2}\cos\omega_1\right\}^2},$$

unde concludimus, si $(\Omega_1-\Omega)$ vel $(R-R_1)>0$ est, Ω minimum et Ω_1 maximum; si vero $(\Omega_1-\Omega)$ vel $(R-R_1)<0$ est, Ω maximum et Ω_1 minimum esse.

Quod radios osculatorios R et R_1 sectionum principalium attinet, ex iis, quae modo diximus, assumptis aequationibus

$$R=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{1}{2}\Omega}$$

et

$$R_1 = \frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{1}{2}\Omega_1}$$

haec sequuntur: si $(R - R_1) > 0$ est, R maximum et R_1 minimum; sin vero $(R - R_1) < 0$ est, R minimum et R_1 maximum est.

Major igitur quisque radius curvaturae sectionum principalium maximum, minor minimum praebet, qua in re non est negligendum, quam significationem calculus differentialis praescribit, eam hic semper maximo et minimo esse subjiciendam.

Conditionem analyticam agnoscendi maximi aut minimi hanc ponere possumus.

Quantitas

$$\frac{1}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\varpi^2}{c^2}} - a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}\left\{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\varpi^2}{c^2}\right\}$$

prout affirmativa aut negativa prodit, vel, quod idem est, prout quantitas

$$a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}\left\{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\varpi^2}{c^2}\right\}^2$$

minor aut major unitate est, Ω et R_1 minima, Ω_1 et R maxima aut Ω et R_1 maxima, Ω_1 et R minima sunt.

Deinde ut definiamus valores quoque angulorum (θ, ω, ϖ) et $(\theta_1, \omega_1, \varpi_1)$, quibus situs sectionum principalium determinatur, quantitates M, M', M'' ex theoria generali notas antea explicari necesse est.

Quae facile exsistunt:

$$M = \frac{2u(2 - b^2\Omega)(2 - c^2\Omega)}{a^2b^2c^2}\,.$$

$$M' = \frac{2v(2 - c^2\Omega)(2 - a^2\Omega)}{a^2b^2c^2}\,,$$

$$M'' = \frac{2w(2 - a^2\Omega)(2 - b^2\Omega)}{a^2b^2c^2}\,;$$

ergo:

$$M^2 + M'^2 + M''^2 = 4\,\frac{(2 - a^2\Omega)(2 - b^2\Omega)(2 - c^2\Omega)}{a^4b^4c^4}$$

$$\times\left\{\left(\frac{u}{2 - a^2\Omega}\right)^2 + \left(\frac{v}{2 - b^2\Omega}\right)^2 + \left(\frac{w}{2 - c^2\Omega}\right)^2\right\}.$$

Itaque ex generali theoria ad angulos eos definiendos obtinemus:

$$\cos\theta = \pm \frac{\frac{u}{2-a^2\Omega}}{\left\{\left(\frac{u}{2-a^2\Omega}\right)^2 + \left(\frac{v}{2-b^2\Omega}\right)^2 + \left(\frac{w}{2-c^2\Omega}\right)^2\right\}^{\frac{1}{2}}},$$

$$\cos\omega = \pm \frac{\frac{v}{2-b^2\Omega}}{\left\{\left(\frac{u}{2-a^2\Omega}\right)^2 + \left(\frac{v}{2-b^2\Omega}\right)^2 + \left(\frac{w}{2-c^2\Omega}\right)^2\right\}^{\frac{1}{2}}},$$

$$\cos\varpi = \pm \frac{\frac{w}{2-c^2\Omega}}{\left\{\left(\frac{u}{2-a^2\Omega}\right)^2 + \left(\frac{v}{2-b^2\Omega}\right)^2 + \left(\frac{w}{2-c^2\Omega}\right)^2\right\}^{\frac{1}{2}}}.$$

Quibus in formulis signa superiora ad superiora, inferiora ad inferiora sunt referenda, sed utra summantur, superiora an inferiora, nihil interest, quippe quibus una modo linea recta per punctum (uvw) ducta determinetur. At satis quidem apparet, formulis illis propter duplicem quantitatis Ω valorem duos pro angulis reddi valores, quibus situs sectionum principalium ellipsoidis determinatus est.

Valores, quos pro $\cos\theta$, $\cos\omega$, $\cos\pi$ dedimus, reales prodire, in iis, quae sequuntur, expositurus sum.

Reales vero prodire valores, si modo Ω quantitas realis est, statim perspicitur.

Eam autem quantitatem realem esse, si expressio

$$\{a^2 + b^2 + c^2 - (u^2 + v^2 + w^2)\}^2 - 4a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}$$

est affirmativa, satis elucet. Quam igitur expressionem esse affirmativam, nunc demonstrabo.

Ex aequatione ellipsoidis sequuntur:

$$u^2 = a^2 - a^2 \cdot \frac{v^2}{b^2} - a^2 \cdot \frac{w^2}{c^2}$$

et

$$\frac{u^2}{a^4} = \frac{1}{a^2} - \frac{v^2}{a^2b^2} - \frac{w^2}{c^2a^2}.$$

Substitutis his valoribus, expressio praecedens transit in hanc:

$$\{b^2 + c^2 + \frac{v^2}{b^2}(a^2 - b^2) - \frac{w^2}{b^2}(c^2 - a^2)\}^2$$

$$-4\{b^2c^2 + c^2 \cdot \frac{v^2}{b^2}(a^2 - b^2) - b^2 . \frac{w^2}{c^2}(c^2 - a^2)\}$$

et inde in hanc formam:

$$\{b^2 + c^2\}^2 + \frac{v^4}{b^4}\{a^2 - b^2\}^2 + \frac{w^4}{c^4}\{c^2 - a^2\}^2$$

$$+ 2\frac{v^2}{b^2}(a^2 - b^2)(b^2 + c^2) - 2\frac{w^2}{c^2}(b^2 + c^2)(c^2 - a^2)$$

$$- 2\frac{v^2}{b^2} \cdot \frac{w^2}{c^2}(c^2 - a^2)(a^2 - b^2)$$

$$- 4\{b^2c^2 + c^2 \cdot \frac{v^2}{b^2}(a^2 - b^2) - b^2 . \frac{w^2}{c^2}(c^2 - a^2)\}.$$

Sunt vero:

$$\{b^2 + c^2\}^2 - 4b^2c^2 = \{b^2 - c^2\}^2,$$

$$2\frac{v^2}{b^2}\{a^2 - b^2\}\{b^2 + c^2\} - 4c^2 . \frac{v^2}{b^2}\{a^2 - b^2\} = 2\frac{v^2}{b^2} \cdot \{a^2 - b^2\}\{b^2 - c^2\},$$

$$-2\frac{w^2}{c^2}\{b^2 + c^2\}\{c^2 - a^2\} + 4b^2 . \frac{w^2}{c^2}\{c^2 - a^2\} = 2\frac{w^2}{c^2} \cdot \{b^2 - c^2\}\{c^2 \quad a^2\};$$

quare valores substituendo pro expressione illa facili calculo obtinemus hanc:

$$\{(b^2 - c^2) + \frac{w^2}{c^2}(c^2 - a^2)\}^2 + \frac{v^4}{b^4}\{a^2 - b^2\}^2$$

$$- 2\frac{v^2}{c^2} \cdot \frac{w^2}{c^2} \cdot \{c^2 - a^2\}\{a^2 - b^2\} + 2\frac{v^2}{b^2}\{a^2 - b^2\}\{b^2 - c^2\}.$$

Primum quantitates *abc* omnes inter se aequales esse si ponamus, expressio ad nihilum reducitur.

Deinde autem fac, ut duae modo quantitates exempli causa *a* et *b* inter se aequales sint, abit in valorem affirmativum

$$\left\{\frac{(c^2 - a^2)(c^2 - w^2)}{c^2}\right\}^2.$$

Tum si quantitates *abc* omnes inter se inaequales sunt, ponere volumus.

$$a^2 > b^2 > c^2$$

esse. Ea conditione sunt:

$$a^2 - b^2 > 0,$$

$$b^2 - c^2 > 0,$$

$$c^2 - a^2 < 0$$

et igitur hoc quoque casu expressionem, de qua quaerimus, affirmativum induere valorem, statim cognoscitur.

Coordinatae denique (X, Y, Z) centrorum curvaturarum sectionum principalium formulis determinantur hisce:

$$X - u =$$

$$-2 \cdot \frac{u b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}}{\left\{ \begin{array}{l} a^2 + b^2 + c^2 - (u^2 + v^2 + w^2) \\ \pm \sqrt{\{a^2 + b^2 + c^2 - (u^2 + v^2 + w^2)\}^2 - 4a^2 b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}} \end{array} \right\}},$$

$$Y - v =$$

$$-2 \cdot \frac{v c^2 a^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}}{\left\{ \begin{array}{l} a^2 + b^2 + c^2 - (u^2 + v^2 + w^2) \\ \pm \sqrt{\{a^2 + b^2 + c^2 - (u^2 + v^2 + w^2)\}^2 - 4a^2 b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}} \end{array} \right\}},$$

$$Z - w =$$

$$-2 \cdot \frac{w a^2 b^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}}{\left\{ \begin{array}{l} a^2 + b^2 + c^2 - (u^2 + v^2 + w^2) \\ \pm \sqrt{\{a^2 + b^2 + c^2 - (u^2 + v^2 + w^2)\}^2 - 4a^2 b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4} \right\}} \end{array} \right\}};$$

ubi signa superiora uni sectioni principali, alteri inferiora valent.

In iis, quae abhinc sequuntur, etiam quaesituri sumus, quos valores et anguli, quibus situs sectionum principalium determinatur, et radii sectionum earum osculatorii induant, si punctum (uvw) in intersectione plani cujusvis coordinatarum et ellipsoidis positum est.

In hoc casu speciali quum multo facilius ex aequationibus, quibus anguli illi ellipsoidi determinantur, quam ex formulis supra

datis angulorum inire possimus rationem, ad angulos definiendos priorem viam inibimus.

Aequationes, quibus in universum anguli illi ellipsoidi determinantur, ex theoria generali sunt:

$$(1)\qquad \frac{u}{a^2}\cos\theta+\frac{v}{b^2}\cos\omega+\frac{w}{c^2}\cos\varpi=0,$$

$$(2)\qquad \frac{u}{a^2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\cos\omega\cos\varpi+\frac{v}{b^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\cos\varpi\cos\theta$$
$$+\frac{w}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\theta\cos\omega=0,$$

$$(3)\qquad \cos\theta^2+\cos\omega^2+\cos\varpi^2=1.$$

In quibus si ponimus

$$u=0,$$

aequationem (2) expletam esse videmus, si

$$\cos\theta=0$$

aut

$$\frac{v}{b^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\cos\varpi+\frac{w}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\omega=0$$

est.

Primum si statuimus

$$\cos\theta=0,$$

ex aequationibus (1) et (3) facili calculo efficiuntur:

$$\cos\omega=\pm\frac{b^2w}{\sqrt{b^4w^2+c^4v^2}},\quad \cos\varpi=\mp\frac{c^2v}{\sqrt{b^4w^2+c^4v^2}};$$

ergo, quia

$$\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}+\frac{\cos\varpi^2}{c^2}=\frac{b^2c^2}{b^4w^2+c^4v^2},$$

$$R=\frac{\{b^4w^2+c^4v^2\}^{\frac{3}{2}}}{b^4c^4}.$$

Porro si ponimus

$$(2)\qquad \frac{v}{b^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\cos\varpi+\frac{w}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\omega=0$$

et valores, quos hoc casu anguli (θ, ω, ϖ) et radius curvaturae R induunt, litteris (θ_1, ω_1, ϖ_1; R_1) significamus, ex aequat.

$$(1) \qquad \frac{v}{b^2}\cos\omega_1 + \frac{w}{c^2}\cos\overline{\omega}_1 = 0$$

et

$$(2) \qquad \frac{v}{b^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\cos\overline{\omega}_1 + \frac{w}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\omega_1 = 0$$

facili calculo obtinemus hanc:

$$\frac{v^2}{b^4}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\cos\omega_1 - \frac{w^2}{c^4}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\omega_1 = 0$$

vel

$$\left\{\frac{v^2}{b^2}(c^2-a^2) - \frac{w^2}{c^2}(a^2-b^2)\right\}\cos\omega_1 = 0,$$

unde sequitur

$$\cos\omega_1 = 0,$$

ideoque ex aequat. (1)

$$\cos\overline{\omega}_1 = 0$$

et ex aequat. (3)

$$\cos\theta_1 = \pm 1.$$

Ergo, quum sit

$$\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} + \frac{\cos\overline{\omega}_1{}^2}{c^2} = \frac{1}{a^2},$$

prodit:

$$R_1 = a^2 \frac{\{b^4 w^2 + c^4 v^2\}^{\frac{3}{2}}}{b^2 c^2}.$$

Deinde in aequationibus supra datis si ponimus

$$v = 0,$$

eodem facili calculo formulas, quibus anguli et radii osculatorii sectionum principalium determinati sunt, invenimus has:

$$\cos\theta = \pm\frac{a^2 w}{\sqrt{c^4 u^2 + a^4 w^2}}, \qquad \cos\omega = 0, \qquad \cos\overline{\omega} = \mp\frac{c^2 u}{\sqrt{c^4 u^2 + a^4 w^2}};$$

ergo, quia

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\overline{\omega}^2}{c^2} = \frac{c^2 a^2}{c^4 u^2 + a^4 w^2},$$

$$R = \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{3}{2}}}{c^4 a^4};$$

et

$$\cos\theta_1 = 0, \quad \cos\omega_1 = \pm 1, \quad \cos\bar{\omega}_1 = 0;$$

itaque, quum

$$\frac{\cos\theta_1^{\,2}}{a^2} + \frac{\cos\omega_1^{\,2}}{b^2} + \frac{\cos\bar{\omega}_1^{\,2}}{c^2} = \frac{1}{b^2}$$

sit,

$$R_1 = b^2 . \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{1}{2}}}{c^2 a^2};$$

tum vero si ponimus

$$w = 0,$$

has:

$$\cos\theta = \pm \frac{a^2 v}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\omega = \mp \frac{b^2 u}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\bar{\omega} = 0;$$

ergo, quum

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} + \frac{\cos\bar{\omega}^2}{c^2} = \frac{a^2 b^2}{a^4 v^2 + b^4 u^2},$$

$$R = \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{3}{2}}}{a^4 b^4};$$

et

$$\cos\theta_1 = 0, \quad \cos\omega_1 = 0, \quad \cos\bar{\omega}_1 = \pm 1;$$

itaque

$$\frac{\cos\theta_1^{\,2}}{a^2} + \frac{\cos\omega_1^{\,2}}{b^2} + \frac{\cos\bar{\omega}_1^{\,2}}{c^2} = \frac{1}{c^2}$$

et

$$R_1 = c^2 . \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{1}{2}}}{a^2 b^2}.$$

In omnibus hisce formulis signa superiora et inferiora in se invicem sunt referenda; sed utra sumantur, nihil interest, quum iis una modo et eadem linea recta determinetur.

Ex valoribus pro angulis datis statim hoc efficitur theorema:

Si punctum (uvw) in intersectione plani cujusvis coordinatarum et ellipsoidis positum est, altera linea, quae sita in plano ellipsoidem in puncto (uvw) tangente alteram sectionem principalem ellipsoidis in puncto illo definit, ei semper axi parallela est, qui ad planum, in quo punctum illud jacet, est perpendicularis; altera vero linea, quae in eodem plano sita alteram sectionem principalem in eo puncto ellipsoidis definit, ad axem illum directa est.

Quo ex theoremate, si modo perpendimus et observamus, normales in vertice ellipsoidis erectas per centrum transire atque in eo, quod supposuimus, coordinatarum systemate vertices*) ea esse puncta, ubi axes coordinatarum ellipsoidem persecant, facili argumentatione efficitur hoc:

Sectiones, quae planis coordinatarum ellipsoidem persecantibus efficiuntur, in vertice ellipsoidis sunt principales.

Ellipsoides rotatione orta.

Circa z axem rotationem factam esse si ponimus, in formulis superioribus $a=b$ ponendum est. Qua re in expressione, quam pro Ω invenimus, si modo aequationis

$$u^2+v^2=\frac{a^2}{c^2}(c^2-w^2)$$

rationem habemus, fiunt:

$$a^2+b^2+c^2-(u^2+v^2+w^2)=\frac{(a^2+c^2)c^2+(a^2-c^2)w^2}{c^2}$$

et

$$a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}=a^2.\frac{c^4+(a^2-c^2)w^2}{c^2}.$$

Quod supra demonstravimus in hoc casu esse

$$\{a^2+b^2+c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}$$

$$=\left\{\frac{(a^2-c^2)(c^2-w^2)}{c^2}\right\}^2,$$

Ω quantitas, si signo utimur superiore, abit in formam:

$$\Omega=\frac{(a^2+c^2)c^2+(a^2-c^2)w^2-(a^2-c^2)(c^2-w^2)}{a^2\{c^4+(a^2-c^2)w^2\}}=\frac{2}{a^2};$$

sin vero inferiore, in hanc:

$$\Omega=\frac{(a^2+c^2)c^2+(a^2-c^2)w^2+(a^2-c^2)(c^2-w^2)}{a^2\{c^4+(a^2-c^2)w^2\}}=\frac{2c^2}{c^4+(a^2-c^2)w^2}.$$

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*) De verticib. ellipsoid. vid. Grunert. geometr. anal. part. II. p. 241. §. 88.

In priori casu, quum sit

$$2 - a^2 \Omega = 0,$$

ex formulis, quas supra in (III) dedimus, cosinus angulorum (θ, ω, ϖ) prodeunt hi:

$$\cos\theta = \frac{\infty}{\infty}, \quad \cos\omega = \frac{\infty}{\infty}, \quad \cos\varpi = 0;$$

in altero vero, si valores, in quos (θ, ω, ϖ) nunc abeunt, per $(\theta_1, \omega_1, \varpi_1)$ significamus, quum sint

$$2 - a^2 \Omega = -2 \cdot \frac{(a^2 - c^2)(c^2 - w^2)}{c^4 + (a^2 - c^2) w^2}$$

et

$$2 - c^2 \Omega = 2 \cdot \frac{w^2 (a^2 - c^2)}{c^4 + (a^2 - c^2) w^2},$$

ratione habita aequationis

$$u^2 + v^2 = \frac{a^2}{c^2} (c^2 - w^2)$$

facili calculo efficiuntur:

$$\cos\theta_1 = \pm \frac{cuw}{\sqrt{(c^2 - w^2)\{c^4 + (a^2 - c^2) w^2\}}},$$

$$\cos\omega_1 = \pm \frac{cvw}{\sqrt{(c^2 - w^2)\{c^4 + (a^2 - c^2) w^2\}}},$$

$$\cos\varpi_1 = \pm \frac{c(c^2 - w^2)}{\sqrt{(c^2 - w^2)\{c^4 + (a^2 - c^2) w^2\}}}.$$

In priori casu cosinus angulorum in valorem ambiguum $\frac{\infty}{\infty}$ abiisse vidimus, unde casum eum excipiendum esse elucet. Ut vero angulorum cognoscamus valores, in iis, quae sequuntur, aequationes ad angulos illos definiendos, quae in hoc casu speciali has praebent formas,

(1) $$\frac{u}{a^2}\cos\theta + \frac{v}{a^2}\cos\omega + \frac{w}{c^2}\cos\varpi = 0,$$

(2) $$\frac{1}{a^2}\left\{\frac{1}{a^2} - \frac{1}{c^2}\right\}\{u\cos\omega - v\cos\theta\}\cos\varpi = 0,$$

(3) $$\cos\theta^2 + \cos\omega^2 + \cos\varpi^2 = 1,$$

separatim persolvemus.

Aequationi (2) satisfaciunt valores

$$\cos\bar{\omega} = 0$$

et

$$u\cos\omega - v\cos\theta = 0.$$

Primum si statuimus

$$\cos\bar{\omega} = 0,$$

ex aequationibus

(1) $$u\cos\theta + v\cos\omega = 0$$

(3) $$\cos\theta^2 + \cos\omega^2 = 1$$

signis superioribus et inferioribus in se invicem relatis sequuntur:

$$\cos\theta = \pm\frac{v}{\sqrt{u^2+v^2}},$$

$$\cos\omega = \mp\frac{u}{\sqrt{u^2+v^2}}.$$

Porro si ponimus

$$u\cos\omega - v\cos\theta = 0$$

et valores, quos in hoc casu anguli $(\theta, \omega, \bar{\omega})$ induunt, per $(\theta_1, \omega_1, \bar{\omega}_1)$ significamus, est:

$$u\cos\omega_1 - v\cos\theta_1 = 0,$$

unde sequitur

$$\cos\omega_1 = \frac{v}{u}\cos\theta_1.$$

Ex aequatione (1) efficitur:

$$\cos\bar{\omega}_1 = -\frac{c^2}{a^2}\cdot\frac{u^2+v^2}{uw}\cos\theta_1$$

et igitur ex (3):

$$\cos\theta_1 = \pm\frac{a^2uw}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}},$$

$$\cos\omega_1 = \pm\frac{a^2vw}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}},$$

$$\cos\bar{\omega}_1 = \mp\frac{c^2(u^2+v^2)}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}};$$

vel

$$\cos\theta_1 = \pm\frac{uw}{c^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}},$$

$$\cos\omega_1 = \pm \frac{vw}{c^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}},$$

$$\cos\overline{\omega}_1 = \mp \frac{u^2+v^2}{a^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}};$$

vel, quum sit

$$u^2+v^2=\frac{a^2}{c^2}(c^2-w^2)$$

et

$$(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}=a^4(c^2-w^2)\cdot\frac{c^4+(a^2-c^2)w^2}{c^2},$$

$$\cos\theta_1=\pm\frac{cuw}{\sqrt{(c^2-w^2)\{c^4+(a^2-c^2)w^2\}}},$$

$$\cos\omega_1=\pm\frac{cvw}{\sqrt{(c^2-w^2)\{c^4+(a^2-c^2)w^2\}}},$$

$$\cos\overline{\omega}_1=\mp\frac{c(c^2-w^2)}{\sqrt{(c^2-w^2)\{c^4+(a^2-c^2)w^2\}}}.$$

In omnibus hisce formulis signa superiora ad superiora, inferiora ad inferiora sunt referenda, sed utra sumantur, nihil interest, quum iis una modo linea recta per punctum (uvw) ducta determinetur.

Ex iis, quae modo invenimus, facili calculo obtinemus:

$$\frac{\cos\theta^2+\cos\omega^2}{a^2}+\frac{\cos\overline{\omega}^2}{c^2}=\frac{1}{a^2}$$

et

$$\frac{\cos\theta_1{}^2+\cos\omega_1{}^2}{a^2}+\frac{\cos\overline{\omega}_1{}^2}{c^2}=\frac{a^2c^2}{c^4(u^2+v^2)+a^4w^2},$$

qua in formula explicanda si modo aequationis

$$c^2(u^2+v^2)=a^2(c^2-w^2)$$

rationem habemus.

Sunt vero:

$$R=\frac{\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2+\cos\omega^2}{a^2}+\frac{\cos\overline{\omega}^2}{c^2}}$$

et

$$R_1 = \frac{\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2+\cos\omega_1{}^2}{a^2}+\frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

ergo valores substituendo:

$$R = \frac{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{1}{2}}}{c^2},$$

$$R_1 = \frac{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{3}{2}}}{a^4c^4};$$

ideoque

$$RR_1 = \frac{\{c^4(u^2+v^2)+a^4w^2\}^2}{a^4c^6}$$

vel

$$RR_1 = a^4c^2\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^2$$

et

$$R+R_1 = \{c^4(u^2+v^2)+a^4w^2\}^{\frac{1}{2}}.\left\{\frac{u^2+v^2}{a^4}+\frac{c^2+w^2}{c^4}\right\}$$

et

$$\frac{1}{R}+\frac{1}{R_1} = a^2c^2.\frac{c^2(a^2+c^2)+w^2(a^2-c^2)}{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{3}{2}}}.$$

In formulis, quas pro ellipsoide generali invenimus, si posuissemus $a=b$, obtinuissemus:

$$R+R_1 = \{2a^2+c^2-(u^2+v^2+w^2)\}\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}},$$

$$\frac{1}{R}+\frac{1}{R_1} = \frac{2a^2+c^2-(u^2+v^2+w^2)}{a^4c^2\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

quas expressiones cum superioribus plane congruere, ratione habita aequationis

$$\frac{u^2+v^2}{a^2}+\frac{w^2}{c^2}=1$$

facile ostendi potest.

Porro ex formulis pro R et R_1 datis spectantes aequationem

$$c^2\{u^2+v^2\} = a^2\{c^2-w^2\}$$

obtinemus:

$$\frac{1}{R}-\frac{1}{R_1}=-a^2c^2\frac{(a^2-c^2)(c^2-w^2)}{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{3}{2}}}$$

et

$$R-R_1=\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}\cdot\frac{(a^2-c^2)(c^2-w^2)}{c^2}.$$

Quum in ellipsoide (c^2-w^2) negativum nunquam esse possit, quantitatibus $(R-R_1)$ et (a^2-c^2) idem signum est, unde assumptis iis, quae in (III) diximus, consequitur:

si $a<c$ est, R minimum et R_1 maximum;
sin vero $a>c$ est, R maximum et R_1 minimum

est.

Hyperboloides à une nappe nominata.

I.

Hyperboloidis à une nappe nominatae aequationem ponentibus generalem

$$u=\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2-1=0,\ ^{*})$$

primae et secundae partiales quotientes differentiales functionis u exsistunt:

$$\frac{du}{dx}=\frac{2x}{a^2},\quad \frac{du}{dy}=\frac{2y}{b^2},\quad \frac{du}{dz}=-\frac{2z}{c^2};$$

$$\frac{d^2u}{dx^2}=\frac{2}{a^2},\quad \frac{d^2u}{dy^2}=\frac{2}{b^2},\quad \frac{d^2u}{dz^2}=-\frac{2}{c^2};$$

$$\frac{d^2u}{dxdy}=0,\quad \frac{d^2u}{dydz}=0,\quad \frac{d^2u}{dzdx}=0.$$

Coordinatae puncti cujusvis in superficie hyperboloidis siti litteris (uvw), coordinatae vero variabiles litteris (xyz) significentur.

Quare ex theoria generali aequationem plani hyperboloidem in puncto (uvw) tangentis obtinemus hanc:

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*) De hyperboloide à une nappe nom. cfr. Grunerti geom. analyt. part. II. §. 77. p. 212.

$$\frac{u}{a^2}(x-u)+\frac{v}{b^2}(y-v)-\frac{w}{c^2}(z-w)=0,$$

vel, quum sit

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2-\left(\frac{w}{c}\right)^2=1,$$

hanc:

$$\frac{ux}{a^2}+\frac{vy}{b^2}-\frac{wz}{c^2}=1.$$

Aequatio plani cujuslibet per punctum (uvw) positi sit:

$$A(x-u)+B(y-v)+C(z-w)=0,$$

et curva, quae plano eo hyperboloidem persecante efficitur, sectio simpliciter vocetur.

Qua in sectione nunc versabimur.

Ex theoria universali aequationes lineae sectionem in puncto (uvw) tangentis invenimus has:

$$-\frac{x-u}{B\frac{w}{c^2}+C\frac{v}{b^2}}=\frac{y-v}{C\frac{u}{a^2}+A\frac{w}{c^2}}=\frac{z-w}{A\frac{v}{b^2}-B\frac{u}{a^2}};$$

hanc plani normalis:

$$-\left\{B\frac{w}{c^2}+C\frac{v}{b^2}\right\}(x-u)+\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}(y-v)+\left\{A\frac{v}{b^2}-B\frac{u}{a^2}\right\}(z-w)=0;$$

has normalis in eodem sectionis puncto erectae:

$$\frac{x-u}{A\left\{A\frac{u}{a^2}+B\frac{v}{b^2}-C\frac{w}{c^2}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2}}$$

$$=\frac{y-v}{B\left\{A\frac{u}{a^2}+B\frac{v}{b^2}-C\frac{w}{c^2}\right\}-\left\{A^2+B^2+C^2\right\}\frac{v}{b^2}}$$

$$=\frac{z-w}{C\left\{A\frac{u}{a^2}+B\frac{v}{b^2}-C\frac{w}{c^2}\right\}+\left\{A^2+B^2+C^2\right\}\frac{w}{c^2}}.$$

Deinde quum quantitates in theoriam generalem brevitatis causa inductae in hoc casu has induant formas:

$$\tfrac{1}{2}U=A\left\{A\frac{u}{a^2}+B\frac{v}{b^2}-C\frac{w}{c^2}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V=B\left\{A\frac{u}{a^2}+B\frac{v}{b^2}-C\frac{w}{c^2}\right\}-\left\{A^2+B^2+C^2\right\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = C\left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\} + \{A^2 + B^2 + C^2\}\frac{w}{c^2},$$

$$\tfrac{1}{4}Q^2 = \left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2 + \left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2$$

$$= (A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\} - \left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\}^2,$$

$$\tfrac{1}{8}\Theta = \frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2,$$

ad coordinatas centri (X, Y, Z) et radium (R) curvaturae in puncto (uvw) sectionis definienda formulas obtinemus hasce:

$$\frac{X - u}{A\left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\} - \{A^2 + B^2 + C^2\}\frac{u}{a^2}}$$

$$= \frac{Y - v}{B\left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\} - \{A^2 + B^2 + C^2\}\frac{v}{b^2}}$$

$$= \frac{Z - w}{C\left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\} + \{A^2 + B^2 + C^2\}\frac{w}{c^2}}$$

$$= \frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\} - \left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2\right]}$$

$$= \frac{\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2 + \left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2\right]}$$

et

$$R =$$

$$\pm\frac{\left[(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\} - \left\{A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2\right]}$$

$$= \pm\frac{\left[\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2 + \left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{c^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2\right]},$$

quibus in duabus formulis signa sumenda sunt ita, ut R sit affirmativum.

Ratione habita aequationis

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 - \left(\frac{w}{c}\right)^2 = 1$$

facili calculo expressionem

$$\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2$$

transmutamus in hanc:

$$\frac{\{Au + Bv + Cw\}^2 - \{A^2a^2 + B^2b^2 - C^2c^2\}}{a^2b^2c^2},$$

quae in formulis datis substitui potest, si in quo casu usui est.

II.

Sectiones normales.

Planum aequatione

$$A(x-u) + B(y-v) + C(z-w) = 0$$

definitum per normalem in puncto (uvw) hyperboloidis erectam transire si ponimus, ex geometriae analyticae principiis debet esse

$$A\frac{u}{a^2} + B\frac{v}{b^2} - C\frac{w}{c^2} = 0.$$

Itaque sunt in hoc casu:

$$\tfrac{1}{2}U = -\{A^2 + B^2 + C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V = -\{A^2 + B^2 + C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = \quad \{A^2 + B^2 + C^2\}\frac{w}{c^2},$$

$$\tfrac{1}{4}Q^2 = (A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}$$

et aequationes, quibus coordinatae centri (XYZ) et radius (R) curvaturae in puncto (uvw) sectionis sunt definita, obtinemus has:

$$X - u = -\frac{u}{a^2} \cdot \frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2},$$

$$Y - v = -\frac{v}{b^2} \cdot \frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2},$$

$$Z - w = \frac{w}{c^2} \cdot \frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2};$$

$$R = \pm \frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}{\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2};$$

ubi signum sumendum est ita, ut R sit positivum.

Quibus ex aequationibus statim consequuntur:

$$X - u = \mp \frac{u}{a^2} \cdot \frac{R}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Y - v = \mp \frac{v}{b^2} \cdot \frac{R}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Z - w = \pm \frac{w}{c^2} \cdot \frac{R}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa superiora aut inferiora valent, prout quantitas

$$\frac{1}{a^2}\left\{B\frac{w}{c^2} + C\frac{v}{b^2}\right\}^2 + \frac{1}{b^2}\left\{C\frac{u}{a^2} + A\frac{w}{c^2}\right\}^2 - \frac{1}{c^2}\left\{A\frac{v}{b^2} - B\frac{u}{a^2}\right\}^2,$$

vel, quod idem est, prout quantitas

$$\{Au + Bv + Cw\}^2 - \{A^2a^2 + B^2b^2 - C^2c^2\}$$

affirmativa aut negativa prodit.

Jam vero perpendiculum ab initio coordinatarum in plano hyperboloidem in puncto (uvw) tangente demissum si littera P significamus, ex geometria analytica est

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}=\frac{1}{P};$$

ergo perpendiculo substituto, si quantitas

$$\{Au+Bv+Cw\}^2-\{A^2a^2+B^2b^2-C^2c^2\}$$

est positiva, formulas obtinemus has:

$$X-u=-\frac{u}{a^2}\cdot PR,$$

$$Y-v=-\frac{v}{b^2}\cdot PR,$$

$$Z-w=\frac{w}{c^2}\cdot PR;$$

unde sequuntur

$$\frac{R\frac{u}{a^2}}{X-u}=\frac{R\frac{v}{b^2}}{Y-v}=-\frac{R\frac{w}{c^2}}{Z-w}=-\frac{1}{P};$$

sin vero quantitas illa in valorem abiit negativum, has:

$$X-u=\frac{u}{a^2}\cdot PR,$$

$$Y-v=\frac{v}{b^2}\cdot PR,$$

$$Z-w=-\frac{w}{c^2}\cdot PR;$$

et ex his:

$$\frac{R\frac{u}{a^2}}{X-u}=\frac{R\frac{v}{b^2}}{Y-v}=-\frac{R\frac{w}{c^2}}{Z-w}=\frac{1}{P}.$$

Deinde ponentibus nobis determinari situm plani sectionis normali in puncto (uvw) hyperboloidis et linea recta in plano hyperboloidem in puncto illo tangente sita, cujus aequationes sunt

$$\frac{x-u}{\cos\theta}=\frac{y-v}{\cos\omega}=\frac{z-w}{\cos\varpi},$$

ubi per $(\theta,\ \omega,\ \varpi)$ designantur anguli 180^0 non superantes, qui linea ea et positivis axium coordinatarum partibus efficiuntur, ex theoria generali consequuntur:

$$X-u=-\frac{\frac{u}{a^2}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}-\frac{\cos\varpi^2}{c^2}},$$

$$Y - v = -\frac{\frac{v}{b^2}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}},$$

$$Z - w = \frac{\frac{w}{c^2}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}};$$

$$R = \pm\frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}},$$

ubi signum sumendum est ita, ut R sit affirmativum.

Alterius sectionis normalis, quam eodem modo, ut praecedentem, sed angulis $(\theta_1, \omega_1, \overline{\omega}_1)$ determinatam esse statuimus, coordinatas centri et radium curvaturae in puncto (uvw) si litteris (X_1, Y_1, Z_1) et (R_1) denotamus, formulas obtinemus has:

$$X_1 - u = -\frac{\frac{u}{a^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} - \frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} - \frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Z_1 - w = \frac{\frac{w}{c^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} - \frac{\cos\overline{\omega}_1{}^2}{c^2}};$$

$$R_1 = \pm\frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} - \frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

ubi signum sumendum est ita, ut R_1 sit positivum.

Quas duas sectiones normales si ad perpendiculum inter se directas esse et brevitatis causa

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2} = \tfrac{1}{2}\Omega,$$

atque

$$\frac{\cos\theta_1{}^2}{a^2}+\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\varpi_1{}^2}{c^2}=\tfrac{1}{2}\Omega_1$$

ponimus, ex theoria generali inter angulos (θ, ω, ϖ) et $(\theta_1, \omega_1, \omega_1)$ hanc habemus aequationem:

$$\tfrac{1}{2}\Omega_1=\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}},$$

unde, quum sit (vide Grunerti Arch. XXVIII. pag. 178.)

$$\left(\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right)\left\{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}\right\}-\left\{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}\right\}$$

$$=\frac{1}{a^2}\left\{\frac{v}{b^2}\cos\varpi+\frac{w}{c^2}\cos\omega\right\}^2+\frac{1}{b^2}\left\{\frac{w}{c^2}\cos\theta+\frac{u}{a^2}\cos\varpi\right\}^2$$

$$-\frac{1}{c^2}\left\{\frac{u}{a^2}\cos\omega-\frac{v}{b^2}\cos\theta\right\}^2$$

ratione habita aequationis

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2-\left(\frac{w}{c}\right)^2=1$$

facili calculo obtinemus hanc:

$$\tfrac{1}{2}\Omega_1=\frac{\{u\cos\theta+v\cos\omega+w\cos\varpi\}^2-\{a^2\cos\theta^2+b^2\cos\omega^2-c^2\cos\varpi^2\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}.$$

Ergo ad quantitates $(X_1, Y_1, Z_1; R_1)$ in hoc casu definiendas hae nobis prodeunt formulae, quibus quantitates illae ab angulis (θ, ω, ϖ) dependentes datae sunt:

$$X_1-u=-\frac{\frac{u}{a^2}}{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}}$$

$$=-\frac{ub^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\{u\cos\theta+b\cos\omega+w\cos\varpi\}^2-\{a^2\cos\theta^2+b^2\cos\omega^2-c^2\cos\varpi^2\}},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= -\frac{vc^2 a^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}^2 - \{a^2\cos\theta^2 + b^2\cos\omega^2 - c^2\cos\overline{\omega}^2\}},$$

$$Z_1 - w = \frac{\frac{w}{c^2}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= \frac{wa^2 b^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}^2 - \{a^2\cos\theta^2 + b^2\cos\omega^2 - c^2\cos\overline{\omega}^2\}};$$

$$R_1 = \pm\frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= \pm\frac{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}{\{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}^2 - \{a^2\cos\theta^2 + b^2\cos\omega^2 - c^2\cos\overline{\omega}^2\}},$$

ubi signa sumenda sunt ita, ut R_1 sit affirmativum.

Inversi igitur radii osculatorii duarum sectionum normalium, quae in puncto (uvw) hyperboloidis inter se sunt perpendiculares, ex formulis superioribus statim sequuntur:

$$\frac{1}{R} = \pm\frac{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \pm\frac{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa superiora aut inferiora sumenda sunt, prout fractionum numeratores, id est, quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ positivae aut negativae prodeunt.

Quae quidem quantitates utrum iisdem semper an diversis signis praedita essent, quaerentibus nobis ea de re apparebat in universum nihil pro certo constitui posse.

Signa vero quantitatum earum quod attinet, duo sunt modo casus, qui per rerum naturam admitti possunt; nam iisdem aut diversis signis quantitates illae praeditae sint, necesse est.

Primum quantitatibus illis eadem signa esse si ponimus, formulas obtinemus has:

$$\frac{1}{R} = \pm \frac{\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \pm \frac{\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa in se invicem referenda, sed superiora aut inferiora sumenda sunt, prout quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ positivae aut negativae prodeunt.

Eadem signorum conditione sunt:

$$\frac{1}{R} + \frac{1}{R_1} = \pm \left[\frac{\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}} \right]$$

$$= \pm \frac{\frac{u^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{v^2}{b^4}\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + \frac{w^2}{c^4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}$$

$$= \pm \frac{-\left(\frac{u}{a}\right)^2 (b^2 - c^2) + \left(\frac{v}{b}\right)^2 (c^2 - a^2) + \left(\frac{w}{c}\right)^2 (a^2 + b^2)}{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

unde ratione habita aequationis

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2-\left(\frac{w}{c}\right)^2=1$$

expressionem obtinemus memorabilem atque elegantem:

$$\frac{1}{R}+\frac{1}{R_1}=\pm\frac{u^2+v^2+w^2-\{a^2+b^2-c^2\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}$$

et

$$\frac{1}{R}-\frac{1}{R_1}=\pm\left[\frac{\frac{\cos 2\theta}{a^2}+\frac{\cos 2\omega}{b^2}-\frac{\cos 2\varpi}{c^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}\right].$$

Sin vero quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ diversis signis praeditae sunt, habemus:

$$\frac{1}{R}=\pm\frac{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}-\frac{\cos\varpi^2}{c^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=\mp\frac{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa in se invicem referenda, sed superiora aut inferiora sumenda sunt, prout $\frac{1}{2}\Omega$ positivum aut negativum est.

Eadem signorum conditione sunt:

$$\frac{1}{R}+\frac{1}{R_1}=\pm\left[\frac{\frac{\cos 2\theta}{a^2}+\frac{\cos 2\omega}{b^2}-\frac{\cos 2\varpi}{c^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}\right],$$

$$\frac{1}{R}-\frac{1}{R_1}=\pm\frac{u^2+v^2+w^2-(a^2+b^2-c^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

III.

Sectiones normales, quae ad curvaturam maximum et minimum praebeant, abhinc occupabimur. Brevitatis causa sectiones eas vocabimus principales.

Sectiones duas normales, quibus occupati sumus, principales esse si statuimus, ex theoria generali inter Ω et Ω_1 has habemus aequationes:

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}\left\{\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right\}-\left\{\frac{u^2}{a^6}+\frac{v^2}{b^6}-\frac{w^2}{c^6}\right\}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}$$

vel, ut jam supra demonstravimus,

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{u^2+v^2+w^2-(a^2+b^2-c^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}$$

et

$$\tfrac{1}{4}\Omega\Omega_1=-\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-1};$$

ex quibus signis superioribus et inferioribus in se invicem relatis sequuntur:

$$\Omega=\frac{\left\{\begin{array}{l}u^2+v^2+w^2-\{a^2+b^2-c^2\}\\ \pm\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}},$$

$$\Omega_1=\frac{\left\{\begin{array}{l}u^2+v^2+w^2-\{a^2+b^2-c^2\}\\ \mp\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}.$$

Ex aequatione

$$\tfrac{1}{4}\Omega\Omega_1=-\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-1}$$

Ω et Ω_1 diversis semper signis praedita esse quum appareat, pro radiis osculatoriis inversis R et R_1 sectionum principalium in puncto (uvw) hyperboloidis expressiones obtinemus has:

$$\frac{1}{R}=\pm\frac{\tfrac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=\mp\frac{\tfrac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa superiora et inferiora in se invicem referenda, sed superiora aut inferiora sumenda sunt, prout $\frac{1}{2}\Omega$ affirmativum aut negativum prodit.

Eadem signorum conditione sunt:

$$\frac{1}{R}+\frac{1}{R_1}=\pm\frac{\frac{1}{2}(\Omega-\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R}-\frac{1}{R_1}=\pm\frac{\frac{1}{2}(\Omega+\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}};$$

ergo, quum facillime intelligatur eadem signorum conditione esse

$$\frac{1}{2}(\Omega-\Omega_1)=\pm\frac{\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}},$$

pro quantitate $\left(\frac{1}{R}+\frac{1}{R_1}\right)$ hanc manifesto obtinemus expressionem:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}};$$

pro $\frac{1}{R}-\frac{1}{R_1}$ vero, si loco $\frac{1}{2}(\Omega+\Omega_1)$ valorem datum substituimus, hanc:

$$\frac{1}{R}-\frac{1}{R_1}=\pm\frac{u^2+v^2+w^2-\{a^2+b^2-c^2\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

ubi, ut jam supra diximus, signum superius aut inferius sumendum est, prout Ω affirmativum aut negativum prodit, vel, quod idem est, prout in expressione pro Ω data signum superius aut inferius sumptum est.

Deinde est

$$\frac{1}{RR_1}=-\frac{\frac{1}{4}\Omega\Omega_1}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}$$

vel, valore pro $\frac{1}{4}\Omega\Omega_1$ substituto,

$$\frac{1}{RR_1}=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-2};$$

et, quum sit

$$R+R_1=\left(\frac{1}{R}+\frac{1}{R_1}\right)RR_1,$$

substituendo prodit:

$$R+R_1=$$

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}\cdot\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}.$$

In formulas superiores si inducimus perpendiculum P, obtinemus:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{P^2}{a^2b^2c^2}\cdot\sqrt{P^2\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2},$$

$$R+R_1=\frac{1}{P^2}\cdot\sqrt{P^2\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2},$$

$$\frac{1}{R}-\frac{1}{R_1}=\pm P^3\cdot\frac{u^2+v^2+w^2-(a^2+b^2-c^2)}{a^2b^2c^2},$$

ubi signa sumenda sunt, ut supra memoravimus, porro

$$\frac{1}{RR_1}=\frac{P^4}{a^2b^2c^2}$$

vel

$$RR_1=\frac{a^2b^2c^2}{P^4}$$

vel

$$a^2b^2c^2=RR_1P^4$$

vel etiam:

$$P=\sqrt[4]{\frac{a^2b^2c^2}{RR_1}}.$$

In expressiones pro $\frac{1}{R}$ et $\frac{1}{R_1}$ supra datas si loco Ω et Ω_1 valores eorum inducimus neque negligimus, quae de signis diximus, quomodo referenda sint ac sumenda, facili argumentatione pro radiis osculatoriis sectionum principalium in puncto (uvw) hyperboloidis expressiones invenimus memorabiles, ubi signa superiora et inferiora in se invicem referenda sunt:

$$\frac{2}{R}=\frac{\left\{\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}} \pm\{u^2+v^2+w^2-(a^2+b^2-c^2)\}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

$$\frac{2}{R_1}=\frac{\left\{\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}} \mp\{u^2+v^2+w^2-(a^2+b^2-c^2)\}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

Sectiones principales ad quantitates Ω et Ω_1, alteram maximum, alteram minimum praebere, theoria generali notum est. Conditiones, quibus Ω et Ω_1 maxima et minima sunt, nunc constituemus.

Sunt:

$$\frac{d^2\Omega}{d\,\theta^2}=2\sin\theta^2\,.\,\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}+\frac{w}{c^2}\cos\omega\right\}^2}\,.\,(\Omega_1-\Omega),$$

$$\frac{d^2\Omega_1}{d\,\theta_1^{\,2}}=2\sin\theta_1^{\,2}\,.\,\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}_1+\frac{w}{c^2}\cos\omega_1\right\}^2}\,.\,(\Omega-\Omega_1),$$

unde sequitur, ut Ω maximum et Ω_1 minimum, si $\Omega-\Omega_1$ est affirmativum; sin negativum, Ω minimum et Ω_1 maximum sit.

Quod Ω et Ω_1 diversis praedita esse signis supra demonstravimus neque igitur $\Omega-\Omega_1$ affirmativum esse potest, nisi Ω est positivum, efficitur, ut affirmativa quaeque quantitas in maximum, negativa in minimum abeat valorem.

Porro etiam nobis sunt determinandi valores angulorum, quibus situs sectionum principalium in puncto (uvw) hyperboloidis definitur. Quem ad finem quantitates M, M', M'' in theoriam generalem brevitatis causa inductae antea explicentur oportet.

Quae facile exsistunt:

$$M=-\frac{2u(2-b^2\,\Omega)(2+c^2\,\Omega)}{a^2b^2c^2},$$

$$M'=-\frac{2v(2+c^2\,\Omega)(2-a^2\,\Omega)}{a^2b^2c^2},$$

$$M'' = -\frac{2w(2-a^2\Omega)(2-b^2\Omega)}{a^2b^2c^2};$$

ergo:

$$M^2 + M'^2 + M''^2 = 4 \cdot \frac{(2-a^2\Omega)^2(2-b^2\Omega)^2(2+c^2\Omega)^2}{a^4b^4c^4}$$

$$\times \left\{ \left(\frac{u}{2-a^2\Omega}\right)^2 + \left(\frac{v}{2-b^2\Omega}\right)^2 + \left(\frac{w}{2+c^2\Omega}\right)^2 \right\};$$

ideoque ex theoria generali ad angulos illos determinandos formulas obtinemus has:

$$\cos\theta = \pm \frac{\dfrac{u}{2-a^2\Omega}}{\left\{ \left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2-b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2 \right\}^{\frac{1}{2}}},$$

$$\cos\omega = \pm \frac{\dfrac{v}{2-b^2\Omega}}{\left\{ \left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2-b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2 \right\}^{\frac{1}{2}}},$$

$$\cos\bar{\omega} = \pm \frac{\dfrac{w}{2+c^2\Omega}}{\left\{ \left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2-b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2 \right\}^{\frac{1}{2}}},$$

ubi signa superiora ad superiora, inferiora ad inferiora referenda sunt, sed utra sumantur, superiora an inferiora, nihil interest, quum iis una modo et eadem linea recta per punctum (uvw) ducta definiatur.

Formulis vero illis, quum expressio pro Ω data duplicis signi causa duos eosque, ut apparet, reales reddat valores, pro angulis duos et reales quidem effici valores, statim perspicitur.

Coordinatae denique (X, Y, Z) centrorum curvaturarum sectionum principalium formulis determinantur his:

$$X - u =$$

$$-2 \cdot \frac{u b^2 c^2 \left\{ \dfrac{u^2}{a^4} + \dfrac{v^2}{b^4} + \dfrac{w^2}{c^4} \right\}}{\left\{ \begin{array}{l} u^2 + v^2 + w^2 - (a^2 + b^2 - c^2) \\ \pm\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2 + 4a^2b^2c^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{w^2}{c^4}\right\}} \end{array} \right\}},$$

$$Y-v=$$

$$-2.\frac{v\,c^2a^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\left\{\begin{array}{l}u^2+v^2+w^2-(a^2+b^2-c^2)\\ \pm\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}},$$

$$Z-w=$$

$$2.\frac{wa^2b^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\left\{\begin{array}{l}u^2+v^2+w^2-(a^2+b^2-c^2)\\ \pm\sqrt{\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}},$$

ubi signa superiora uni sectioni principali, alteri inferiora valent.

Ad ultimum nobis etiam liceat quaerere, quos in valores et anguli, quibus situs sectionum principalium determinatur, et radii sectionum earum osculatorii abeant, si punctum (uvw) in intersectione plani cujusvis coordinatarum et hyperboloidis positum est.

Majores quum nobis sint ineundae rationes, si angulorum valores ex formulis datis quam si ex aequationibus determinamus, quibus in universum anguli illi hyperboloidi à une nappe nominatae definiuntur, in iis, quae sequuntur, ad angulos definiendos viam inibimus posteriorem.

Aequationes, quibus in universum anguli illi hyperboloidi à une nappe nominatae determinantur, ex theoria generali prodeunt hae:

(1) $$\frac{u}{a^2}\cos\theta+\frac{v}{b^2}\cos\omega-\frac{w}{c^2}\cos\varpi=0,$$

(2) $$\frac{u}{a^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\cos\omega\cos\varpi-\frac{v}{b^2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)\cos\varpi\cos\theta-\frac{w}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos\theta\cos\omega=0,$$

(3) $$\cos\theta^2+\cos\omega^2+\cos\varpi^2=1.$$

Quas aequationes si comparamus cum iis, quibus anguli illi ellipsoidi determinantur, ex illis prodire videmus, si in illis pro c^2 quantitas $(-c^2)$ ponitur. Ergo in formulis, quibus anguli illi ellipsoidi determinati sunt, quantitate $(-c^2)$ pro c^2 substituta assecuturos nos esse formulas, quibus anguli illi hyperboloidi à

une nappe nominatae determinati sunt, statim elucet. Itaque angulos et radium curvaturae alterius sectionis principalis si litteris $(\theta, \omega, \bar{\omega}; R)$, alterius litteris $(\theta_1, \omega_1, \bar{\omega}_1; R_1)$ significamus ac primum ponimus

$$u = 0,$$

obtinemus formulas has:

$$\cos\theta = 0, \quad \cos\omega = \pm \frac{b^2 w}{\sqrt{b^4 w^2 + c^4 v^2}}, \quad \cos\bar{\omega} = \pm \frac{c^2 v}{\sqrt{b^4 w^2 + c^4 v^2}};$$

$$R = \frac{\{b^4 w^2 + c^4 v^2\}^{\frac{3}{2}}}{b^4 c^4};$$

$$\cos\theta_1 = \pm 1, \quad \cos\omega_1 = 0, \quad \cos\bar{\omega}_1 = 0;$$

$$R_1 = a^2 \frac{\{b^4 w^2 + c^4 v^2\}^{\frac{1}{2}}}{b^2 c^2};$$

sin vero ponimus

$$v = 0,$$

has:

$$\cos\theta = \pm \frac{a^2 w}{\sqrt{c^4 u^2 + a^4 w^2}}, \quad \cos\omega = 0, \quad \cos\bar{\omega} = \pm \frac{c^2 u}{\sqrt{c^4 u^2 + a^4 w^2}};$$

$$R = \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{3}{2}}}{c^4 a^4};$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = \pm 1, \quad \cos\bar{\omega}_1 = 0;$$

$$R_1 = b^2 . \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{1}{2}}}{c^2 a^2};$$

denique, si ponimus

$$w = 0,$$

has:

$$\cos\theta = \pm \frac{a^2 v}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\omega = \mp \frac{b^2 u}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\bar{\omega} = 0;$$

$$R = \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{3}{2}}}{a^4 b^4}.$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = 0, \quad \cos\bar{\omega}_1 = \pm 1;$$

$$R_1 = c^2 \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{1}{2}}}{a^2 b^2}.$$

Ex valoribus pro angulis datis idem, quod ellipsoidi, efficitur theorema:

Si punctum (uvw) in intersectione plani cujusvis coordinatarum et hyperboloidis à une nappe nominatae positum est, altera linea, quae sita in plano hyperboloidem in puncto (uvw) tangente alteram sectionem principalem hyperboloidis in puncto illo determinat, ei semper axi coordinatarum parallela est, qui ad planum, in quo punctum jacet, est perpendicularis; altera vero linea, quae in eodem plano sita alteram sectionem principalem in eo puncto hyperboloidis determinat, ad axem illum directa est.

Quum hyperboloidis quoque à une nappe nominatae normales in vertice erectae per centrum transeant atque in nostro coordinatarum systemate vertices*) ea sint puncta, ubi axes coordinatarum hyperboloidem persecant, ex theoremate illo efficitur hoc:

Sectiones, quae planis coordinatarum hyperboloidem à une nappe nominatam persecantibus efficiuntur, in vertice hyperboloidis sunt principales.

Hyperboloides à une nappe rotatione descripta.

In generali hyperboloidis à une nappe nominatae aequatione si ponimus $a=b$, aequationem obtinemus hyperboloidis à une nappe, quae rotatione hyperbolae circa z axem descripta est.

Itaque, si ponimus $a=b$ atque aequationis modo

$$u^2+v^2=\frac{a^2}{c^2}\{c^2+w^2\}$$

rationem habemus, facili calculo fiunt:

$$u^2+v^2+w^2-(a^2+b^2-c^2)=\frac{c^4+(a^2+c^2)w^2-a^2c^2}{c^2},$$

$$a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}=\frac{a^2}{c^2}\cdot\{c^4+(a^2+c^2)w^2\},$$

$$\{u^2+v^2+w^2-(a^2+b^2-c^2)\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}$$

$$=\left\{\frac{c^4+(a^2+c^2)w^2+a^2c^2}{c^2}\right\}^2;$$

*) De verticib. hyperboloid. à une nappe nominatae vid. Grunerti geometr. analyt. part. II. §. 88. p. 241 et 242.

ergo, signo si utimur superiore,

$$\Omega = \frac{2}{a^2},$$

sin inferiore

$$\Omega = -\frac{2c^2}{c^4 + (a^2 + c^2)w^2}.$$

Quare ex formulis supra datis, quum in priori casu sit

$$2 - a^2\Omega = 0,$$

cosinus angulorum $(\theta, \omega, \overline{\omega})$ prodeunt hi:

$$\cos\theta = \frac{\infty}{\infty}, \quad \cos\omega = \frac{\infty}{\infty}, \quad \cos\overline{\omega} = 0;$$

angulorum vero $(\theta_1, \omega_1, \overline{\omega}_1)$, quibus alterius sectionis principalis situs determinetur, quum in altero casu sint

$$2 - a^2\Omega = 2 \cdot \frac{(a^2 + c^2)(c^2 + w^2)}{c^4 + (a^2 + c^2)w^2}$$

et

$$2 + c^2\Omega = 2 \cdot \frac{(a^2 + c^2)w^2}{c^4 + (a^2 + c^2)w^2},$$

ratione habita aequationis

$$u^2 + v^2 = \frac{a^2}{c^2}(c^2 + w^2)$$

facili calculo hi:

$$\cos\theta_1 = \pm\frac{cuw}{\sqrt{(c^2 + w^2)\{c^4 + (a^2 + c^2)w^2\}}},$$

$$\cos\omega_1 = \pm\frac{cvw}{\sqrt{(c^2 + w^2)\{c^4 + (a^2 + c^2)w^2\}}},$$

$$\cos\overline{\omega}_1 = \pm\frac{c(c^2 + w^2)}{\sqrt{(c^2 + w^2)\{c^4 + (a^2 + c^2)w^2\}}};$$

quas formulas, quum sint

$$(c^2 + w^2) = \frac{c^2}{a^2}\{u^2 + v^2\},$$

$$(c^2 + w^2)\{c^4 + (a^2 + c^2)w^2\} = \frac{c^2}{a^4}(u^2 + v^2)\{c^4(u^2 + v^2) + a^4w^2\},$$

facile transmutamus in has:

$$\cos\theta_1 = \pm \frac{a^2uw}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}},$$

$$\cos\omega_1 = \pm \frac{a^2vw}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}},$$

$$\cos\overline{\omega}_1 = \pm \frac{c^2(u^2+v^2)}{\sqrt{(u^2+v^2)\{c^4(u^2+v^2)+a^4w^2\}}};$$

vel in has:

$$\cos\theta_1 = \pm \frac{uw}{c^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}},$$

$$\cos\omega_1 = \pm \frac{vw}{c^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}},$$

$$\cos\overline{\omega}_1 = \pm \frac{u^2+v^2}{a^2\sqrt{(u^2+v^2)\left\{\frac{u^2+v^2}{a^4}+\frac{w^2}{c^4}\right\}}}.$$

Quum in priori casu cosinus angulorum θ et ω in formam ambiguam $\frac{\infty}{\infty}$ abiissent, in iis, quae sequuntur, ut valores cognoscamus angulorum, ex aequationibus, quibus anguli illi hyperboloidi à une nappe rotatione descriptae determinantur, rationem inibimus valorum, quos anguli θ et ω induunt, si est $\cos\overline{\omega} = 0$.

Aequationes illae sunt:

$$(1) \qquad \frac{u}{a^2}\cos\theta + \frac{v}{b^2}\cos\omega - \frac{w}{c^2}\cos\overline{\omega} = 0,$$

$$(2) \qquad (u\cos\omega - v\cos\theta)\cos\overline{\omega} = 0,$$

$$(3) \qquad \cos\theta^2 + \cos\omega^2 + \cos\overline{\omega}^2 = 1,$$

unde, posito $\cos\overline{\omega} = 0$, facili calculo obtinemus:

$$\cos\theta = \pm \frac{v}{\sqrt{u^2+v^2}},$$

$$\cos\omega = \mp \frac{u}{\sqrt{u^2+v^2}}.$$

In hisce et superioribus formulis signa superiora et inferiora in se invicem referenda sunt, sed utra sumantur, nihil refert, quum iis una modo et eadem linea recta per punctum (uvw) ducta definiatur.

Ex formulis superioribus sequuntur:

$$\frac{\cos\theta^2 + \cos\omega^2}{a^2} - \frac{\cos\overline{\omega}^2}{c^2} = \frac{1}{a^2}$$

et aequationis

$$c^2(u^2+v^2) = a^2(c^2+w^2)$$

ratione habita:

$$\frac{\cos{\theta_1}^2 + \cos{\omega_1}^2}{a^2} - \frac{\cos{\overline{\omega}_1}^2}{c^2} = -\frac{a^2c^2}{c^4(u^2+v^2)+a^4w^2}.$$

Sunt vero:

$$R = \pm \frac{\left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2 + \cos\omega^2}{a^2} - \frac{\cos\overline{\omega}^2}{c^2}}$$

et

$$R_1 = \pm \frac{\left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos{\theta_1}^2 + \cos{\omega_1}^2}{a^2} - \frac{\cos{\overline{\omega}_1}^2}{c^2}},$$

ubi signa sumenda sunt ita, ut radii sint affirmativi; ergo, substitutione facta,

$$R = a^2 \cdot \left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}} = \frac{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{1}{2}}}{c^2}$$

et

$$R_1 = a^2c^2 \cdot \left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}} = \frac{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{3}{2}}}{a^4c^4};$$

unde facili calculo efficiuntur:

$$RR_1 = a^4c^2 \cdot \left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^2 = \frac{\{c^4(u^2+v^2)+a^4w^2\}^2}{a^4c^6},$$

$$R + R_1 = \frac{(a^2+c^2)(u^2+v^2)}{a^2} \cdot \left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}},$$

$$\frac{1}{R} + \frac{1}{R_1} = \frac{(a^2+c^2)(c^2+w^2)}{a^4c^4\left\{\frac{u^2+v^2}{a^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}} = \frac{a^2c^2(a^2+c^2)(c^2+w^2)}{\{c^4(u^2+v^2)+a^4w^2\}^{\frac{3}{2}}}.$$

Hyperboloides à deux nappes nominata.

I.

Hyperboloidis à deux nappes nominatae aequatio sit generalis

$$u = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 - 1 = 0. \text{ *)}$$

Functione u partialiter ad x, y, z differentiata primae et secundae partiales quotientes differentiales functionis u prodeunt hae:

$$\frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = -\frac{2y}{b^2}, \quad \frac{du}{dz} = -\frac{2z}{c^2};$$

$$\frac{d^2u}{dx^2} = \frac{2}{a^2}, \quad \frac{d^2u}{dy^2} = -\frac{2}{b^2}, \quad \frac{d^2u}{dz^2} = -\frac{2}{c^2};$$

$$\frac{d^2u}{dxdy} = 0, \quad \frac{d^2u}{dydz} = 0, \quad \frac{d^2u}{dzdx} = 0.$$

Coordinatas puncti cujusvis in superficie hyperboloidis siti litteris (uvw), variabiles vero litteris (xyz) si denotamus, ex theoria generali aequationem plani hyperboloidem in puncto (uvw) tangentis invenimus hanc:

$$\frac{u}{a^2}(x-u) - \frac{v}{b^2}(y-v) - \frac{w}{c^2}(z-w) = 0$$

vel, quum sit

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 - \left(\frac{w}{c}\right)^2 = 1,$$

hanc:

$$\frac{ux}{a^2} - \frac{vy}{b^2} - \frac{wz}{c^2} = 1.$$

Sit porro aequatio plani cujusvis per punctum (uvw) positi

$$A(x-u) + B(y-v) + C(z-w) = 0,$$

et curva, quae plano eo hyperboloidem persecante efficitur, brevitatis causa sectio nominetur.

Qua sectione in iis, quae sequuntur, occupabimur.

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*) De hyperboloide à deux nappes nominat. vide Grunerti geom. analyt. part. II. §. 77. p. 214.

Theoria universali aequationes lineae sectionem in puncto (uvw) tangentis obtinemus has:

$$\frac{x-u}{B\frac{w}{c^2}-C\frac{v}{b^2}}=-\frac{y-v}{C\frac{u}{a^2}+A\frac{w}{c^2}}=\frac{z-w}{A\frac{v}{b^2}+B\frac{u}{a^2}};$$

hanc plani normalis:

$$(B\frac{w}{c^2}-C\frac{v}{b^2})(x-u)-(C\frac{u}{a^2}+A\frac{w}{c^2})(y-v)+(A\frac{v}{b^2}+B\frac{u}{a^2})(z-w)=0;$$

has normalis in illo sectionis puncto erectae:

$$\frac{x-u}{A\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}-\{A^2+B^2+C^2\}\frac{u}{a^2}}$$

$$=\frac{y-v}{B\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}+\{A^2+B^2+C^2\}\frac{v}{b^2}}$$

$$=\frac{z-w}{C\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}+\{A^2+B^2+C^2\}\frac{w}{c^2}}.$$

Deinde quum sint quantitates in theoriam generalem brevitatis causa inductae:

$$\tfrac{1}{2}U=A\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}-\{A^2+B^2+C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V=B\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}+\{A^2+B^2+C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W=C\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}+\{A^2+B^2+C^2\}\frac{w}{c^2};$$

$$\tfrac{1}{4}Q^2=\{A\frac{v}{b^2}+B\frac{u}{a^2}\}^2+\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2+\{C\frac{u}{a^2}+A\frac{w}{c^2}\}^2$$

$$=(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}-\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\}^2,$$

$$\tfrac{1}{8}\Theta=\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2-\frac{1}{b^2}\{C\frac{u}{a^2}+A\frac{w}{c^2}\}^2-\frac{1}{c^2}\{A\frac{v}{b^2}+B\frac{u}{a^2}\}^2;$$

formulas, quibus coordinatae centri (X, Y, Z) atque radius curvaturae (R) in puncto (uvw) sectionis sunt definita, obtinemus has:

$$\frac{X-u}{A\left\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2}}$$

$$=\frac{Y-v}{B\left\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\right\}+\left\{A^2+B^2+C^2\right\}\frac{v}{b^2}}$$

$$=\frac{Z-w}{C\left\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\right\}+\left\{A^2+B^2+C^2\right\}\frac{w}{c^2}}$$

$$=\frac{\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2+\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2+\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2-\frac{1}{b^2}\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2-\frac{1}{c^2}\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2\right]}$$

$$=\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}-\left\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\right\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2-\frac{1}{b^2}\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2-\frac{1}{c^2}\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2\right]}$$

et

$$R=$$

$$\pm\frac{\left[\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2+\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2+\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{c^2}\right\}^2-\frac{1}{b^2}\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2-\frac{1}{c^2}\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2\right]},$$

vel

$$R=$$

$$\pm\frac{\left[(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}-\left\{A\frac{u}{a^2}-B\frac{v}{b^2}-C\frac{w}{c^2}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2-\frac{1}{b^2}\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2-\frac{1}{c^2}\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2\right]}$$

ubi signa ita sumenda sunt, ut R fiat affirmativum.

Expressionem

$$\frac{1}{a^2}\left\{B\frac{w}{c^2}-C\frac{v}{b^2}\right\}^2-\frac{1}{b^2}\left\{C\frac{u}{a^2}+A\frac{w}{c^2}\right\}^2-\frac{1}{c^2}\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}^2$$

ratione habita aequationis

$$\left(\frac{u}{a}\right)^2-\left(\frac{v}{b}\right)^2-\left(\frac{w}{c}\right)^2=1$$

facili calculo transformamus in hanc:

$$\frac{A^2a^2 - B^2b^2 - C^2c^2 - \{Au + Bv + Cw\}^2}{a^2b^2c^2},$$

quae in formulis superioribus potest substitui, si quando ex usu est

II.

Jam vero sectiones agemus normales.

Planum aequatione

$$A(x-u) + B(y-v) + C(z-w) = 0$$

definitum per normalem in puncto (uvw) hyperboloidis erectam transire et igitur sectionem describere normalem si, ponimus ex geometriae analyticae principiis

$$A\frac{u}{a^2} - B\frac{v}{b^2} - C\frac{w}{c^2} = 0$$

sit necesse est.

Itaque sunt sectioni normali:

$$\tfrac{1}{2}U = -\{A^2 + B^2 + C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V = \quad \{A^2 + B^2 + C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = \quad \{A^2 + B^2 + C^2\}\frac{w}{c^2};$$

$$\tfrac{1}{4}Q^2 = (A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\};$$

ergo:

$$X - u = -\frac{u}{a^2}\cdot\frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2} - C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} + A\frac{w}{c^2}\}^2 - \frac{1}{c^2}\{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2},$$

$$Y - v = \quad \frac{v}{b^2}\cdot\frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2} - C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} + A\frac{w}{c^2}\}^2 - \frac{1}{c^2}\{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2},$$

$$Z - w = \frac{w}{c^2}\cdot\frac{(A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{\frac{1}{a^2}\{B\frac{w}{c^2} - C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} + A\frac{w}{c^2}\}^2 - \frac{1}{c^2}\{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2};$$

$$R = \pm \frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}{\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2-\frac{1}{b^2}\{C\frac{u}{a^2}+A\frac{w}{c^2}\}^2-\frac{1}{c^2}\{A\frac{v}{b^2}+B\frac{u}{a^2}\}^2},$$

ubi signum sumendum est ita, ut R sit positivum.

Quibus e formulis sequuntur:

$$X-u = \mp\frac{u}{a^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Y-v = \pm\frac{v}{b^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$Z-w = \pm\frac{w}{c^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}};$$

ubi signa superiora aut inferiora patet esse sumenda, prout quantitas

$$\frac{1}{a^2}\{B\frac{w}{c^2}-C\frac{v}{b^2}\}^2-\frac{1}{b^2}\{C\frac{u}{a^2}+A\frac{w}{c^2}\}^2-\frac{1}{c^2}\{A\frac{v}{b^2}+B\frac{u}{a^2}\}^2,$$

vel etiam, ut ex (I) elucet, prout quantitas

$$A^2a^2-B^2b^2-C^2c^2-\{Au+Bv+Cw\}^2$$

positiva aut negativa prodit.

Denotante P perpendiculum ab initio coordinatarum systematis in plano hyperboloidem à deux nappes in puncto (uvw) tangente demissum, ex geometria analytica est nominatam

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}} = \frac{1}{P}.$$

Quo perpendiculo substituto formulae superiores, si quantitas

$$A^2a^2-B^2b^2-C^2c^2-\{Au+Bv+Cw\}^2$$

est positiva, formas induunt has:

$$X-u = -\frac{u}{a^2}\cdot PR,$$

$$Y-v = \quad\frac{v}{b^2}\cdot PR,$$

$$Z-w = \quad\frac{w}{c^2}\cdot PR;$$

ergo:

$$-\frac{R\frac{u}{a^2}}{X-u}=\frac{R\frac{v}{b^2}}{Y-v}=\frac{R\frac{w}{c^2}}{Z-w}=\frac{1}{P};$$

sin negativa, has:

$$X-u=\frac{u}{a^2}\cdot PR,$$

$$Y-v=-\frac{v}{b^2}\cdot PR,$$

$$Z-w=-\frac{w}{c^2}\cdot PR;$$

ergo:

$$\frac{R\frac{u}{a^2}}{X-u}=-\frac{R\frac{v}{b^2}}{Y-v}=-\frac{R\frac{w}{c^2}}{Z-w}=\frac{1}{P}.$$

Jam vero linea recta in plano hyperboloidem in puncto (uvw) tangente sita, cujus aequationes sunt:

$$\frac{x-u}{\cos\theta}=\frac{y-v}{\cos\omega}=\frac{z-w}{\cos\overline{\omega}},$$

situm sectionis normalis definiri si ponimus, theoria generali aequationes, quibus coordinatae centri (XYZ) atque radius R curvaturae in puncto (uvw) sectionis sunt definita, prodeunt hae:

$$X-u=-\frac{\frac{u}{a^2}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}},$$

$$Y-v=\frac{\frac{v}{b^2}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}},$$

$$Z-w=\frac{\frac{w}{c^2}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}};$$

$$R=\pm\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}};$$

ubi signum ita sumendum est, ut R fiat affirmativum.

Deinde alteram lineam rectam in plano hyperboloidem in puncto (uvw) tangente per punctum (uvw) ductam esse atque ea linea, cujus aequationes sint:

$$\frac{x-u}{\cos\theta_1}=\frac{y-v}{\cos\omega_1}=\frac{z-w}{\cos\overline{\omega}_1},$$

alterius sectionis normalis situm determinari si statuimus, ad coordinatas centri (X_1, Y_1, Z_1) et radium R_1 curvaturae hujus sectionis definienda formulas obtinemus hasce:

$$X_1-u=-\frac{\frac{u}{a^2}}{\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Y_1-v=\frac{\frac{v}{b^2}}{\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\overline{\omega}_1{}^2}{c^2}},$$

$$Z_1-w=\frac{\frac{w}{c^2}}{\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\overline{\omega}_1{}^2}{c^2}};$$

$$R_1=\pm\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\overline{\omega}_1{}^2}{c^2}};$$

ubi signum sumendum est ita, ut R_1 sit affirmativum.

Quae duae sectiones normales si inter se sunt perpendiculares atque brevitatis causa

$$\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}=\tfrac{1}{2}\Omega,$$

et

$$\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}-\frac{\cos\overline{\omega}_1{}^2}{c^2}=\tfrac{1}{2}\Omega_1$$

ponitur, hanc inter angulos $(\theta, \omega, \overline{\omega})$ et $(\theta_1, \omega_1, \overline{\omega}_1)$ ex theoria generali obtinemus aequationem:

$$\tfrac{1}{2}\Omega_1=\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\sin\overline{\omega}^2}{c^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}.$$

Quum autem sit (vid. Grunerti Arch. XXVIII. pag. 178.)

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}\left\{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}\right\}-\left\{\frac{u^2}{a^6}-\frac{v^2}{b^6}-\frac{w^2}{c^6}\right\}$$

$$=\frac{1}{a^2}\left\{\frac{v}{b^2}\cos\varpi-\frac{w}{c^2}\cos\omega\right\}^2-\frac{1}{b^2}\left\{\frac{w}{c^2}\cos\theta+\frac{u}{a^2}\cos\varpi\right\}^2$$

$$-\frac{1}{c^2}\left\{\frac{u}{a^2}\cos\omega+\frac{v}{b^2}\cos\theta\right\}^2,$$

ratione habita aequationis

$$\left(\frac{u}{a}\right)^2-\left(\frac{v}{b}\right)^2-\left(\frac{w}{c}\right)^2=1$$

facili calculo expressionem pro $\frac{1}{2}\Omega_1$ modo datam mutamus in

$$\frac{a^2\cos\theta^2-b^2\cos\omega^2-c^2\cos\varpi^2-\{u\cos\theta+v\cos\omega+w\cos\varpi\}^2}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}.$$

Ergo in hoc casu ad quantitates (X_1, Y_1, Z_1; R_1) definiendas obtinemus formulas, quibus quantitates ab angulis (θ, ω, ϖ) dependentes datae sunt:

$$X_1-u=-\frac{\frac{u}{a^2}}{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}}$$

$$=-\frac{ub^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{a^2\cos\theta^2-b^2\cos\omega^2-c^2\cos\varpi^2-\{u\cos\theta+v\cos\omega+w\cos\varpi\}^2},$$

$$Y_1-v=\frac{\frac{v}{b^2}}{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\sin\varpi^2}{c^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}-\frac{w^2}{c^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}}$$

$$=\frac{vc^2a^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{a^2\cos\theta^2-b^2\cos\omega^2-c^2\cos\varpi^2-\{u\cos\theta+v\cos\omega+w\cos\varpi\}^2},$$

$$Z_1 - w = \frac{\frac{w}{c^2}}{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\varpi^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= \frac{wa^2b^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}}{a^2\cos\theta^2 - b^2\cos\omega^2 - c^2\cos\varpi^2 - \{u\cos\theta + v\cos\omega + w\cos\varpi\}^2};$$

$$R_1 = \pm \frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\varpi^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}$$

$$= \pm \frac{a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}{a^2\cos\theta^2 - b^2\cos\omega^2 - c^2\cos\varpi^2 - \{u\cos\theta + v\cos\omega + w\cos\varpi\}^2},$$

ubi signa sumenda sunt, ut R_1 sit positivum.

Duabus igitur sectionibus normalibus, quae in puncto (uvw) inter se sunt perpendiculares, inversi radii osculatorii prodeunt hi:

$$\frac{1}{R} = \pm \frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\varpi^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \pm \frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\varpi^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa superiora aut inferiora sumenda sunt, prout fractionum numeratores, id est, prout quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$, affirmativae aut negativae prodeunt.

Disquisitiones nostrae, quas de quantitatibus $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ instituimus, utrum iisdem an diversis signis praeditae essent oporteret, non eo duxerunt, ut in universum aliquid pro certo a nobis posset constitui.

Primum igitur quantitatibus $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ eadem signa esse si ponimus, signis superioribus et inferioribus in se invicem relatis formulas obtinemus has:

$$\frac{1}{R} = \pm \frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \pm \frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa superiora aut inferiora sumenda sunt, prout quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ positivae aut negativae sunt.

Eadem signorum conditione est:

$$\frac{1}{R} + \frac{1}{R_1} = \pm \left\{ \frac{\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}} \right\}$$

$$= \pm \frac{\left(\frac{u}{a}\right)^2 \{-b^2 - c^2\} - \left(\frac{v}{b}\right)^2 (a^2 - c^2) - \left(\frac{w}{c}\right)^2 (a^2 - b^2)}{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

unde ratione habita aequationis

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 - \left(\frac{w}{c}\right)^2 = 1$$

facili calculo obtinemus expressionem memorabilem:

$$\frac{1}{R} + \frac{1}{R_1} = \pm \frac{a^2 - b^2 - c^2 - \{u^2 + v^2 + w^2\}}{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

Ex aequatione hyperboloidis

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 - \left(\frac{w}{c}\right)^2 = 1$$

$a^2 - u^2$ in valorem affirmativum nunquam abire posse quum eluceat ac manifestum sit, differentiam

$$a^2 - b^2 - c^2 - \{u^2 + v^2 + w^2\}$$

negativam semper esse, ergo quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ una affirmativos induere valores nullo modo posse efficitur.

Si igitur quantitates $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ iisdem sunt praedita signis, semper ponenda sunt:

$$\frac{1}{R} = -\frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = -\frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ergo:

$$\frac{1}{R} + \frac{1}{R_1} = \frac{u^2 + v^2 + w^2 - (a^2 - b^2 - c^2)}{a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

$$\frac{1}{R} - \frac{1}{R_1} = -\left\{\frac{\frac{\cos 2\theta}{a^2} - \frac{\cos 2\omega}{b^2} - \frac{\cos 2\overline{\omega}}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}} + \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}\right\}.$$

Sin vero $\frac{1}{2}\Omega$ et $\frac{1}{2}\Omega_1$ diversis signis praedita sunt, formulas obtinemus has:

$$\frac{1}{R} = \pm\frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = \mp\frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\sin\overline{\omega}^2}{c^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

ubi signa in se invicem referenda, sed superiora aut inferiora sumenda sunt, prout $\frac{1}{2}\Omega$ positivum aut negativum prodit, et eadem signorum conditione:

$$\frac{1}{R} + \frac{1}{R_1} = \pm\left\{\frac{\frac{\cos 2\theta}{a^2} - \frac{\cos 2\omega}{b^2} - \frac{\cos 2\overline{\omega}}{c^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}} + \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6} - \frac{w^2}{c^6}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}}\right\},$$

$$\frac{1}{R}-\frac{1}{R_1}=\pm\frac{a^2-b^2-c^2-\{u^2+v^2+w^2\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}}.$$

III.

Sectionibus principalibus, hoc est, sectionibus, quae ad curvaturam maximum et minimum praebeant, nunc occupabimur.

Sectiones duas normales, quas tractavimus, in puncto (uvw) hyperboloidis esse principales si ponimus, ex theoria universali sit nesesse est:

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}\left\{\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right\}-\left\{\frac{u^2}{a^6}-\frac{v^2}{b^6}-\frac{w^2}{c^6}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}$$

$$=\frac{a^2-b^2-c^2-\{u^2+v^2+w^2\}}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}$$

et

$$\tfrac{1}{4}\,\Omega\Omega_1=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-1};$$

unde signis superioribus et inferioribus in se invicem relatis consequuntur:

$$\Omega=\frac{\left\{\begin{array}{l}a^2-b^2-c^2-\{u^2+v^2+w^2\}\\ \pm\sqrt{\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}},$$

$$\Omega_1=\frac{\left\{\begin{array}{l}a^2-b^2-c^2-\{u^2+v^2+w^2\}\\ \mp\sqrt{\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}}{a^2\,b^2\,c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}.$$

Ex aequatione

$$\tfrac{1}{4}\,\Omega\Omega_1=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-1}$$

Ω et Ω_1 iisdem signis praedita esse quum appareat et supra demonstratum sit, differentiam

$$a^2-b^2-c^2-\{u^2+v^2+w^2\}$$

negativam esse, sectionibus principalibus quantitates Ω et Ω_1 negativas semper prodire patet.

Sunt igitur:

$$\frac{1}{R} = -\frac{\frac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = -\frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}};$$

ergo:

$$\frac{1}{R}+\frac{1}{R_1} = -\frac{\frac{1}{2}(\Omega+\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{RR_1} = \frac{\frac{1}{4}\Omega\Omega_1}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}},$$

$$R+R_1 = -\frac{\frac{1}{2}(\Omega+\Omega_1)}{\frac{1}{4}\Omega\Omega_1}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}},$$

vel, valoribus pro $\frac{1}{2}(\Omega+\Omega_1)$ et $\frac{1}{4}\Omega\Omega_1$ substitutis,

$$\frac{1}{R}+\frac{1}{R_1} = \frac{u^2+v^2+w^2-\{a^2-b^2-c^2\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

$$R+R_1 = \{u^2+v^2+w^2-(a^2-b^2-c^2)\}\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{1}{2}},$$

$$\frac{1}{RR_1} = \frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{-2}}{a^2b^2c^2};$$

vel, si inducimus perpendiculum P,

$$\frac{1}{R}+\frac{1}{R_1} = P^3\,\frac{u^2+v^2+w^2-(a^2-b^2-c^2)}{a^2b^2c^2},$$

$$R+R_1 = \frac{u^2+v^2+w^2-\{a^2-b^2-c^2\}}{P},$$

$$\frac{1}{RR_1} = \frac{P^4}{a^2\,b^2\,c^2}$$

vel

$$RR_1 = \frac{a^2 b^2 c^2}{P^4}$$

vel

$$a^2 b^2 c^2 = RR_1 P^4$$

vel etiam:

$$P = \sqrt[4]{\frac{a^2 b^2 c^2}{RR_1}}.$$

Jam vero in formulis

$$\frac{1}{R} = -\frac{\frac{1}{2}\Omega}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = -\frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}},$$

si loco Ω et Ω_1 valores datos substituimus, pro radiis osculatoriis sectionum principalium expressiones obtinemus memorabiles hasce:

$$\frac{2}{R} = \frac{\left\{\begin{array}{l} u^2 + v^2 + w^2 - \{a^2 - b^2 - c^2\} \\ \mp \sqrt{\{u^2 + v^2 + w^2 - (a^2 - b^2 - c^2)\}^2 - 4a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}} \end{array}\right\}}{a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

$$\frac{2}{R_1} = \frac{\left\{\begin{array}{l} u^2 + v^2 + w^2 - \{a^2 - b^2 - c^2\} \\ \pm \sqrt{\{u^2 + v^2 + w^2 - (a^2 - b^2 - c^2)\}^2 - 4a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}} \end{array}\right\}}{a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{3}{2}}},$$

ubi signa superiora ad superiora, inferiora ad inferiora referenda sunt.

Mentionem etiam facturi sumus, ex formula

$$RR_1 = a^2b^2c^2 \cdot \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^2,$$

quam si dividimus per

$$R = -\frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2}},$$

pro R_1 expressionem memorabilem effici hanc:

$$R_1 = -a^2b^2c^2 \left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}^{\frac{3}{2}} \left\{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}-\frac{\cos\overline{\omega}^2}{c^2}\right\}.$$

Sectiones principales ad quantitates Ω et Ω_1 maximum alteram, alteram minimum praebere novimus. Nunc vero conditiones, quibus Ω et Ω_1 maxima et minima sunt, constituemus.

Sunt:

$$\frac{d^2\Omega}{d\,\theta^2} = 2\sin\theta^2 \cdot \frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}-\frac{w}{c^2}\cos\omega\right\}^2}\cdot(\Omega_1-\Omega),$$

$$\frac{d^2\Omega_1}{d\theta_1{}^2} = 2\sin\theta_1{}^2 \cdot \frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}}{\left\{\frac{v}{b^2}\cos\overline{\omega}_1-\frac{w}{c^2}\cos\omega_1\right\}^2}\cdot(\Omega-\Omega_1),$$

unde ex principiis calculi differentialis concludimus, Ω maximum et Ω_1 minimum esse, si $\Omega-\Omega_1$ est positivum; sin negativum, Ω minimum et Ω_1 maximum esse.

In iis, quae sequuntur, valores definiemus angulorum, quibus situs sectionum principalium determinatur.

Quem ad finem quantitates M, M', M'' in theoriam generalem brevitatis causa inductae antea explicentur oportet.

Quae facile exsistunt:

$$M = \frac{2u(2+b^2\Omega)(2+c^2\Omega)}{a^2b^2c^2},$$

$$M' = \frac{2v(2+c^2\Omega)(2-a^2\Omega)}{a^2b^2c^2},$$

$$M'' = \frac{2w(2-a^2\Omega)(2+b^2\Omega)}{a^2b^2c^2};$$

itaque

$$M^2+M'^2+M''^2 = 4\cdot\frac{(2-a^2\Omega)(2+b^2\Omega)(2+c^2\Omega)}{a^4b^4c^4}$$

$$\times\left\{\left(\frac{u}{2-a^2\Omega}\right)^2+\left(\frac{v}{2+b^2\Omega}\right)^2+\left(\frac{w}{2+c^2\Omega}\right)^2\right\};$$

ergo ex theoria generali:

$$\cos\theta = \pm \frac{\dfrac{u}{2-a^2\Omega}}{\left\{\left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2+b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2\right\}^{\frac{1}{2}}},$$

$$\cos\omega = \pm \frac{\dfrac{v}{2+b^2\Omega}}{\left\{\left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2+b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2\right\}^{\frac{1}{2}}},$$

$$\cos\overline{\omega} = \pm \frac{\dfrac{w}{2+c^2\Omega}}{\left\{\left(\dfrac{u}{2-a^2\Omega}\right)^2 + \left(\dfrac{v}{2+b^2\Omega}\right)^2 + \left(\dfrac{w}{2+c^2\Omega}\right)^2\right\}^{\frac{1}{2}}};$$

ubi signa superiora ad superiora, inferiora ad inferiora referenda sunt, sed utra sumantur, nihil interest, quum iis una modo et eadem linea recta per punctum (uvw) ducta definiatur.

Formulis vero illis, quum expressio pro Ω data duos pro Ω reddat valores, duos eosque, si Ω quantitas est realis, reales quoque pro angulis reddi valores, satis apparet.

Ω quantitatem esse realem, nunc demonstrabo.

Ex expressione pro Ω supra data Ω esse reale, si differentia

$$\{a^2 - b^2 - c^2 - (u^2 + v^2 + w^2)\}^2 - 4a^2b^2c^2\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}\right\}$$

est positiva, cognoscitur.

Ergo differentiam esse positivam, expositurus sum.

Ex generali hyperboloidis à deux nappes nominatae aequatione sequuntur:

$$u^2 = a^2\left\{1 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2\right\},$$

$$\frac{u^2}{a^4} = \frac{1}{a^2}\left\{1 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2\right\}.$$

Quibus valoribus pro u^2 et $\dfrac{u^2}{a^4}$ substitutis differentia illa abit in

$$\left\{b^2 + c^2 + \frac{v^2}{b^2}(a^2 + b^2) + \frac{w^2}{c^2}(c^2 + a^2)\right\}^2$$

$$-4\left\{b^2c^2 + \frac{v^2}{b^2}\cdot c^2(a^2 + b^2) + \frac{w^2}{c^2}\cdot b^2(c^2 + a^2)\right\}$$

et inde in

$$(b^2+c^2)^2+\frac{v^4}{b^4}(a^2+b^2)^2+\frac{w^4}{c^4}(c^2+a^2)^2$$

$$+2\frac{v^2}{b^2}(a^2+b^2)(b^2+c^2)+2\frac{w^2}{c^2}(b^2+c^2)(c^2+a^2)$$

$$+2\frac{v^2}{b^2}\cdot\frac{w^2}{c^2}(c^2+a^2)(a^2+b^2)$$

$$-4\left\{b^2c^2+\frac{v^2}{b^2}\cdot c^2(a^2+b^2)+\frac{w^2}{c^2}\cdot b^2(c^2+a^2)\right\},$$

et, si conjungimus, quae conjungenda sunt, in hanc:

$$\left\{(b^2-c^2)-\frac{w^2}{c^2}(c^2+a^2)\right\}^2+\frac{v^4}{b^4}(a^2+b^2)^2$$

$$+2\frac{v^2}{b^2}\cdot\frac{w^2}{c^2}(c^2+a^2)(a^2+b^2)+2\frac{v^2}{b^2}(a^2+b^2)(b^2-c^2)$$

vel etiam in hanc formam:

$$\left\{(b^2-c^2)+\frac{v^2}{b^2}(a^2+b^2)^2\right\}+\frac{w^4}{c^4}(c^2+a^2)^2$$

$$+2\frac{v^2}{b^2}\cdot\frac{w^2}{c^2}(c^2+a^2)(a^2+b^2)-2\frac{w^2}{c^2}(c^2+a^2)(b^2-c^2).$$

Si b^2-c^2 est positivum, forma priori, sin negativum, altera differentiam, de qua quaerimus, esse affirmativam perspicitur.

Est igitur Ω semper reale et formulae superiores pro angulis reales reddunt valores.

Coordinatae porro centrorum curvaturarum sectionum principalium theoria generali determinantur formulis hisce:

$$X-u=$$

$$-2.\frac{u\,b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\left\{\begin{matrix}a^2-b^2-c^2-\{u^2+v^2+w^2\}\\ \pm\sqrt{\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{matrix}\right\}},$$

$$Y-v=$$

$$2.\frac{v\,c^2a^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\left\{\begin{matrix}a^2-b^2-c^2-\{u^2+v^2+w^2\}\\ \pm\sqrt{\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{matrix}\right\}},$$

$$Z - w =$$

$$2.\quad \frac{wa^2b^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}{\left\{\begin{array}{l} a^2-b^2-c^2-\{u^2+v^2+w^2\} \\ \pm\sqrt{\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}}\end{array}\right\}},$$

ubi signa superiora uni sectioni principali, alteri inferiora valent.

Denique etiam quaesituri sumus, quos valores et anguli et radii sectionum principalium osculatorii induant, si punctum (uvw) in intersectione plani cujusvis coordinatarum et hyperboloidis positum est.

Eadem, quam bis jam attulimus, causa hic quoque non ex formulis supra datis sed ex aequationibus, quibus in universum anguli illi hyperboloidi à deux nappes nominatae definiuntur, angulorum determinantur valores.

Aequationes illae sunt:

$$(1)\qquad \frac{u}{a^2}\cos\theta - \frac{v}{b^2}\cos\omega - \frac{w}{c^2}\cos\bar{\omega} = 0,$$

$$(2)\qquad \frac{u}{a^2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\cos\omega\cos\bar{\omega} - \frac{v}{b^2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)\cos\bar{\omega}\cos\theta$$
$$+\frac{w}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cos\bar{\omega}\cos\theta = 0,$$

$$(3)\qquad \cos\theta^2 + \cos\omega^2 + \cos\bar{\omega}^2 = 1.$$

Ex aequatione hyperboloidis

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 - \left(\frac{w}{c}\right)^2 = 1$$

$u = 0$ non esse ponendum, ergo plano (yz) hyperboloidem non persecari apparet.

Sin vero ponimus

$$v = 0,$$

quod per aequationem illam licet, aequationi (2) satisfaciunt valores

$$\cos\omega = 0$$

et

$$\frac{u}{a^2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\cos\bar{\omega} + \frac{w}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cos\theta = 0.$$

Primum si ponimus

$$\cos\omega = 0,$$

ex aequationibus (1) et (3) facili calculo efficiuntur:

$$\cos\theta = \pm \frac{a^2 w}{\sqrt{c^4 u^2 + a^4 w^2}},$$

$$\cos\overline{\omega} = \pm \frac{c^2 u}{\sqrt{c^4 u^2 + a^4 w^2}}.$$

Deinde si ponimus

$$\frac{u}{a^2}\left\{\frac{1}{b^2} - \frac{1}{c^2}\right\}\cos\overline{\omega} + \frac{w}{c^2}\left\{\frac{1}{a^2} + \frac{1}{b^2}\right\}\cos\theta = 0,$$

et valores, in quos hoc casu anguli (θ, ω, $\overline{\omega}$) et radius curvaturae R abeunt, litteris (θ_1, ω_1, $\overline{\omega}_1$; R_1) denotamus, ex aequationibus

$$(1) \qquad \frac{u}{a^2}\cos\theta_1 - \frac{w}{c^2}\cos\overline{\omega}_1 = 0,$$

$$(2) \qquad \frac{u}{a^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)\cos\overline{\omega}_1 + \frac{w}{c^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\cos\theta_1 = 0$$

facili calculo obtinemus hanc:

$$\left\{\frac{u^2}{a^4}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + \frac{w^2}{c^4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\}\cos\theta_1 = 0,$$

unde sequitur

$$\cos\theta_1 = 0,$$

ergo ex aequatione (1)

$$\cos\omega_1 = 0$$

et ex aeq. (3)

$$\cos\overline{\omega}_1 = \pm 1.$$

Porro sunt:

$$\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2} = -\frac{c^2 a^2}{c^4 u^2 + a^4 w^2},$$

$$\frac{\cos\theta_1{}^2}{a^2} - \frac{\cos\omega_1{}^2}{b^2} - \frac{\cos\overline{\omega}_1{}^2}{c^2} = -\frac{1}{b^2};$$

ergo:

$$R = \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{3}{2}}}{c^4 a^4},$$

$$R_1 = b^2 . \frac{\{c^4 u^2 + a^4 w^2\}^{\frac{1}{2}}}{c^2 a^2}.$$

Denique si ponimus

$$w = 0,$$

ex aequationibus datis eodem facili calculo obtinemus:

$$\cos\theta = \pm \frac{a^2 v}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\omega = \pm \frac{b^2 u}{\sqrt{a^4 v^2 + b^4 u^2}}, \quad \cos\overline{\omega} = 0;$$

ergo, quum sit

$$\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} - \frac{\cos\overline{\omega}^2}{c^2} = -\frac{a^2 b^2}{a^4 v^2 + b^4 u^2},$$

$$R = \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{3}{2}}}{a^4 b^4};$$

et

$$\cos\theta_1 = 0, \quad \cos\omega_1 = 0, \quad \cos\overline{\omega}_1 = \pm 1;$$

ergo, quia

$$\frac{\cos{\theta_1}^2}{a^2} - \frac{\cos{\omega_1}^2}{b^2} - \frac{\cos{\overline{\omega}_1}^2}{c^2} = -\frac{1}{b^2},$$

$$R_1 = c^2 \frac{\{a^4 v^2 + b^4 u^2\}^{\frac{1}{2}}}{a^2 b^2}.$$

Ex valoribus pro angulis datis hoc efficitur theorema:

Si punctum (uvw) in intersectione plani cujusvis coordinatarum et hyperboloidis à deux nappes nominatae positum est, altera linea, quae sita in plano hyperboloidem in puncto (uvw) tangente alteram sectionem principalem hyperboloidis in puncto (uvw) determinat, ei semper axi coordinatarum parallela est, qui ad planum, in quo punctum jacet, est perpendicularis; altera vero linea, quae in eodem plano sita alteram sectionem principalem in puncto (uvw) hyperboloidis determinat, ad axem illum directa est.

Quo ex theoremate, si modo observamus, normales in vertice hyperboloidis à deux nappes nominatae per centrum transire atque vertices*) ea esse puncta, ubi axes coordinatarum hyperboloidem persecant, facili cogitatione sequitur hoc:

Sectiones, quae planis coordinatarum hyperboloidem à deux nappes nominatam persecantibus efficiuntur, in vertice hyperboloidis sunt principales.

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*) De verticib. hyperboloid. à deux nappes nominatae vid. Grunerti geom. analyt. part. II. §. 88. p. 242.

Hyperboloides à deux nappes rotatione descripta.

In generali hyperboloidis à deux nappes nominatae aequatione si ponimus $b=c$, aequationem obtinemus hyperboloidis à deux nappes, quae rotatione hyperbolae circa x-axem descripta est.

Si igitur ponimus $b=c$ et aequationis modo

$$v^2+w^2=\frac{b^2}{a^2}(u^2-a^2)$$

rationem habemus, facili calculo fiunt:

$$a^2-b^2-c^2-\{u^2+v^2+w^2\}=\frac{a^2(a^2-b^2)-u^2(a^2+b^2)}{a^2},$$

$$a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}=\frac{b^2}{a^2}\{-a^4+(a^2+b^2)u^2\}.$$

Ex expressione, in quam supra differentiam

$$\{a^2-b^2-c^2-(u^2+v^2+w^2)\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{w^2}{c^4}\right\}$$

transmutavimus, differentiam illam, si est $b=c$, abire in

$$\left\{\frac{(v^2+w^2)(a^2+b^2)}{b^2}\right\}^2$$

vel in

$$\left\{\frac{(u^2-a^2)(a^2+b^2)}{a^2}\right\}^2$$

statim cognoscitur.

Ergo, signo si utimur superiore, fit:

$$\Omega=\frac{a^2(a^2-b^2)-u^2(a^2+b^2)+(u^2-a^2)(a^2+b^2)}{b^2\{-a^4+(a^2+b^2)u^2\}},$$

$$=-\frac{2a^2}{u^2(a^2+b^2)-a^4},$$

sin inferiore:

$$\Omega=\frac{a^2(a^2-b^2)-u^2(a^2+b^2)-(u^2-a^2)(a^2+b^2)}{b^2\{-a^4+(a^2+b^2)u^2\}}=-\frac{2}{b^2}.$$

In priori igitur casu, quum sint:

$$2-a^2\Omega=2.\frac{(a^2+b^2)u^2}{u^2(a^2+b^2)-a^4},$$

$$2+b^2\Omega=2.\frac{(a^2+b^2)(u^2-a^2)}{u^2(a^2+b^2)-a^4},$$

ex formulis supra datis cosinus angulorum $(\theta,\ \omega,\ \overline{\omega})$ facili calculo prodeunt hi:

$$\cos\theta=\pm\frac{a(u^2-a^2)}{\sqrt{(u^2-a^2)\{u^2(a^2+b^2)-a^4\}}},$$

$$\cos\omega=\pm\frac{auv}{\sqrt{(u^2-a^2)\{u^2(a^2+b^2)-a^4\}}},$$

$$\cos\overline{\omega}=\pm\frac{auw}{\sqrt{(u^2-a^2)\{u^2(a^2+b^2)-a^4\}}};$$

vel, quia

$$u^2-a^2=\frac{a^2}{b^2}\{v^2+w^2\},$$

$$u^2\{a^2+b^2\}-a^4=\frac{a^4\{v^2+w^2\}+b^4u^2}{b^2};$$

$$\cos\theta=\pm\frac{a^2(v^2+w^2)}{\sqrt{(v^2+w^2)\{a^4(v^2+w^2)+b^4u^2\}}},$$

$$\cos\omega=\pm\frac{b^2uv}{\sqrt{(v^2+w^2)\{a^4(v^2+w^2)+b^4u^2\}}},$$

$$\cos\overline{\omega}=\pm\frac{b^2uw}{\sqrt{(v^2+w^2)\{a^4(v^2+w^2)+b^4u^2\}}};$$

vel etiam:

$$\cos\theta=\pm\frac{v^2+w^2}{b^2\sqrt{(v^2+w^2)\left\{\frac{u^2}{a^4}+\frac{v^2+w^2}{b^4}\right\}}},$$

$$\cos\omega=\pm\frac{uv}{a^2\sqrt{(v^2+w^2)\left\{\frac{u^2}{a^4}+\frac{v^2+w^2}{b^4}\right\}}},$$

$$\cos\overline{\omega}=\pm\frac{uw}{a^2\sqrt{(v^2+w^2)\left\{\frac{u^2}{a^4}+\frac{v^2+w^2}{b^4}\right\}}};$$

in altero vero, quum sint

$$2-a^2\Omega=2\,.\frac{b^2-a^2}{a^2}$$

et

$$2+b^2\Omega=0,$$

cosinus angulorum (θ_1, ω_1, $\overline{\omega}_1$), quibus alterius sectionis principalis situs determinetur, prodeunt hi:

$$\cos\theta_1=0,\quad \cos\omega_1=\frac{\infty}{\infty},\quad \cos\overline{\omega}_1=\frac{\infty}{\infty}.$$

Hoc casu quod in determinandis angulorum ω_1 et $\overline{\omega}_1$ cosinibus ad formam ambiguam $\frac{\infty}{\infty}$ pervenimus et igitur in medio relicti sumus, in iis, quae sequuntur, ut angulorum cognoscamus valores, ex aequationibus, quibus anguli illi hyperboloidi à deux nappes rotatione descriptae definiuntur, rationem inibimus valorum, in quos cosinus angulorum θ_1 et ω_1 abeunt, si est $\cos\theta_1=0$.

Aequationes illae sunt:

$$\text{(1)}\qquad \frac{u}{a^2}\cos\theta_1-\frac{v}{b^2}\cos\omega_1-\frac{w}{c^2}\cos\overline{\omega}_1=0,$$

$$\text{(2)}\qquad \{v\cos\overline{\omega}_1-w\cos\omega_1\}\cos\theta_1=0,$$

$$\text{(3)}\qquad \cos\theta_1{}^2+\cos\omega_1{}^2+\cos\overline{\omega}_1{}^2=1;$$

ex quibus, posito $\cos\theta_1=0$, facili calculo obtinemus:

$$\cos\omega_1=\pm\frac{v}{\sqrt{v^2+w^2}},$$

$$\cos\overline{\omega}_1=\mp\frac{w}{\sqrt{v^2+w^2}}.$$

In omnibus hisce formulis signa superiora et inferiora in se invicem referenda sunt, sed utra sumantur, nihil interest, quum iis una modo et eadem linea recta definiatur.

Ex formulis superioribus invenimus:

$$\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2+\cos\overline{\omega}^2}{b^2}=-\frac{a^2b^2}{a^4(v^2+w^2)+b^4u^2}$$

et

$$\frac{\cos\theta_1{}^2}{c^2}-\frac{\cos\omega_1{}^2+\cos\overline{\omega}_1{}^2}{b^2}=-\frac{1}{b^2};$$

ergo, quum sint

$$R=-\frac{\left\{\frac{u^2}{a^4}+\frac{v^2+w^2}{b^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2+\cos\overline{\omega}^2}{b^2}},$$

$$R_1 = -\frac{\left\{\frac{u^2}{a^4} + \frac{v^2+w^2}{b^4}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2}{a^2} - \frac{\cos\omega_1{}^2 + \cos\overline{\omega}_1{}^2}{b^2}},$$

substitutione facta fiunt:

$$R = a^2b^2 \cdot \left\{\frac{u^2}{a^4} + \frac{v^2+w^2}{b^4}\right\}^{\frac{3}{2}} = \frac{\{a^4(v^2+w^2) + b^4u^2\}^{\frac{3}{2}}}{a^4b^4},$$

$$R_1 = b^2 \cdot \left\{\frac{u^2}{a^4} + \frac{v^2+w^2}{b^4}\right\}^{\frac{1}{2}} = \frac{\{a^4(v^2+w^2) + b^4u^2\}^{\frac{1}{2}}}{a^2};$$

unde facili calculo obtinemus:

$$RR_1 = a^2b^4 \left\{\frac{u^2}{a^4} + \frac{v^2+w^2}{b^4}\right\}^2 = \frac{\{a^4(v^2+w^2) + b^4u^2\}^2}{a^6b^4},$$

$$R + R_1 = \{a^4(v^2+w^2) + b^4u^2\}^{\frac{1}{2}} \cdot \left\{\frac{a^2+u^2}{a^4} + \frac{v^2+w^2}{b^4}\right\},$$

$$\frac{1}{R} + \frac{1}{R_1} = a^2 \cdot \frac{a^4(v^2+w^2) + b^4(u^2+a^2)}{\{a^4(v^2+w^2) + b^4u^2\}^{\frac{3}{2}}}.$$

Paraboloides elliptica.

I.

Paraboloidis ellipticae aequatio sit in universum

$$u = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \pm \frac{2z}{c} = 0. \text{ *)}$$

In iis, quae sequuntur, aequationibus ac formulis signa duplicia, quae ad signa in aequatione paraboloidis ellipticae non sunt referenda, ut errores praecaveamus, uncis includantur. Signa vero, quae uncis inclusa non sunt, superiora ad superius, inferiora ad inferius, quod in aequatione paraboloidis ellipticae invenitur, signum referenda sunt.

Functione u partialiter ad x, y, z differentiata primae et secundae partiales quotientes differentiales functionis u prodeunt hae:

$$\frac{du}{dx} = \frac{2x}{a^2}, \qquad \frac{du}{dy} = \frac{2y}{b^2}, \qquad \frac{du}{dz} = \pm\frac{2}{c};$$

*) De paraboloide elliptica cfr. Grunerti geom. analyt. part. II. §. 79. pag. 217.

$$\frac{d^2u}{dx^2}=\frac{2}{a^2},\qquad \frac{d^2u}{dy^2}=\frac{2}{b^2},\qquad \frac{d^2u}{dz^2}=0;$$

$$\frac{d^2u}{dxdy}=0,\qquad \frac{d^2u}{dydz}=0,\qquad \frac{d^2u}{dzdx}=0.$$

Porro si coordinatas puncti cujusvis in superficie paraboloidis ellipticae siti litteris (uvw), variabiles vero litteris (xyz) denotamus, ex theoria generali aequationem plani paraboloidem ellipticam in puncto (uvw) tangentis obtinemus hanc:

$$\frac{u}{a^2}(x-u)+\frac{v}{b^2}(y-v)\pm\frac{1}{c}(z-w)=0$$

vel, quum sit

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2\pm\frac{2w}{c}=0,$$

hanc:

$$\frac{ux}{a^2}+\frac{vy}{b^2}\pm\frac{z}{c}=\mp\frac{w}{c}.$$

Per punctum (uvw) paraboloidis ellipticae planum aequatione

$$A(x-u)+B(y-v)+C(z-w)=0$$

definitum ponemus et curvam, quae plano eo paraboloidem persecante efficitur, brevitatis causa sectionem appellabimus.

Qua sectione nunc occupabimur.

Theoria generali aequationes lineae sectionem in puncto (uvw) tangentis prodeunt hae:

$$\frac{x-u}{\pm B\frac{1}{c}-C\frac{v}{b^2}}=\frac{y-v}{C\frac{u}{a^2}\mp A\frac{1}{c}}=\frac{z-w}{A\frac{v}{b^2}-B\frac{u}{a^2}};$$

haec plani normalis:

$$\left\{\pm B\frac{1}{c}-C\frac{v}{b^2}\right\}(x-u)+\left\{C\frac{u}{a^2}\mp A\frac{1}{c}\right\}(y-v)+\left\{A\frac{v}{b^2}-B\frac{u}{a^2}\right\}(z-w)=0;$$

hae normalis in puncto (uvw) sectionis erectae:

$$\frac{x-u}{A\left\{A\frac{u}{a^2}+B\frac{v}{b^2}\pm C\frac{1}{c}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2}}$$

$$=\frac{y-v}{B\left\{A\frac{u}{a^2}+B\frac{v}{b^2}\pm C\frac{1}{c}\right\}-\left\{A^2+B^2+C^2\right\}\frac{v}{b^2}}$$

$$=\frac{z-w}{C\left\{A\frac{u}{a^2}+B\frac{v}{b^2}\pm C\frac{1}{c}\right\}\mp\left\{A^2+B^2+C^2\right\}\frac{1}{c}}.$$

Porro quum quantitates brevitatis causa in theoriam generalem inductae exsistant:

$$\tfrac{1}{2}U = A\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} - \{A^2 + B^2 + C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V = B\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} - \{A^2 + B^2 + C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = C\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} \mp \{A^2 + B^2 + C^2\}\frac{1}{c};$$

$$\tfrac{1}{4}Q^2 = \{A\frac{v}{b^2} - B\frac{u}{a^2}\}^2 + \{\pm B\frac{1}{c} - C\frac{v}{b^2}\}^2 + \{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2,$$

vel

$$\tfrac{1}{4}Q^2 = (A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\} - \{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\}^2,$$

$$\tfrac{1}{8}\Theta = \frac{1}{a^2}\{\pm B\frac{1}{c} - C\frac{v}{b^2}\}^2 + \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2;$$

ad coordinatas centri (XYZ) ac radium (R) curvaturae in puncto (uvw) sectionis definienda obtinemus formulas hasce:

$$\frac{X - u}{A\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} - \{A^2 + B^2 + C^2\}\frac{u}{a^2}}$$

$$= \frac{Y - v}{B\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} - \{A^2 + B^2 + C^2\}\frac{v}{b^2}}$$

$$= \frac{Z - w}{C\{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\} \mp \{A^2 + B^2 + C^2\}\frac{1}{c}}$$

$$= \frac{\{A\frac{v}{b^2} - B\frac{u}{a^2}\}^2 + \{\pm B\frac{1}{c} - C\frac{v}{b^2}\}^2 + \{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\{\pm B\frac{1}{c} - C\frac{v}{b^2}\}^2 + \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]}$$

$$= \frac{(A^2 + B^2 + C^2)\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\} - \{A\frac{u}{a^2} + B\frac{v}{b^2} \pm C\frac{1}{c}\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\{\pm B\frac{1}{c} - C\frac{v}{b^2}\}^2 + \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]}$$

et

$$R=\frac{\left[\left\{A\frac{v}{b^2}-B\frac{u}{a^2}\right\}^2+\left\{\pm B\frac{1}{c}-C\frac{v}{b^2}\right\}^2+\left\{C\frac{u}{a^2}\mp A\frac{1}{c}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\cdot\left[\frac{1}{a^2}\left\{\pm B\frac{1}{c}-C\frac{v}{b^2}\right\}^2+\frac{1}{b^2}\left\{C\frac{u}{a^2}\mp A\frac{1}{c}\right\}^2\right]}$$

$$=\frac{\left[(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}-\left\{A\frac{u}{a^2}+B\frac{v}{b^2}\pm C\frac{1}{c}\right\}^2\right]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\cdot\left[\frac{1}{a^2}\left\{\pm B\frac{1}{c}-C\frac{v}{b^2}\right\}^2+\frac{1}{b^2}\left\{C\frac{u}{a^2}\mp A\frac{1}{c}\right\}^2\right]}.$$

Expressionem pro $\frac{1}{8}\Theta$ supra datam ratione habita aequationis

$$\left(\frac{u}{a}\right)^2+\left(\frac{v}{b}\right)^2\pm\frac{2w}{c}=0$$

facili calculo transmutamus in hanc:

$$\frac{A^2a^2+B^2b^2\mp 2Cc\{Au+Bv+Cw\}}{a^2b^2c^2},$$

quae in formulis superioribus potest substitui, si quando in rem est.

II.

Sectiones normales nunc tractabimus.

Planum aequatione

$$A(x-u)+B(y-v)+C(z-w)=0$$

definitum per normalem in puncto (uvw) paraboloidis ellipticae erectam transire si ponimus, ex analyticae geometriae principiis

$$A\frac{u}{a^2}+B\frac{v}{b^2}\pm C\frac{1}{c}=0$$

sit necesse est.

Itaque sunt in hoc casu:

$$\tfrac{1}{2}U=-\{A^2+B^2+C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V=-\{A^2+B^2+C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W=\mp\{A^2+B^2+C^2\}\frac{1}{c};$$

$$\tfrac{1}{4}Q^2=(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\};$$

atque coordinatae centri et radius curvaturae sectionis normalis formulis determinantur hisce:

$$X-u=-\frac{u}{a^2}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2},$$

$$Y-v=-\frac{v}{b^2}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2},$$

$$Z-w=\mp\frac{1}{c}\cdot\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2};$$

$$R=\frac{(A^2+B^2+C^2)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}{\frac{1}{a^2}\{\pm B\frac{1}{c}-C\frac{v}{b^2}\}^2+\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2};$$

e quibus statim consequuntur:

$$X-u=-\frac{u}{a^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$Y-v=-\frac{v}{b^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$Z-w=\mp\frac{1}{c}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ita ut hoc casu etiam sit:

$$\frac{R\frac{u}{a^2}}{X-u}=\frac{R\frac{v}{b^2}}{Y-v}=\pm\frac{R\frac{1}{c}}{Z-w}=-\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}.$$

Littera P si denotamus perpendiculum ab initio coordinatarum systematis in plano paraboloidem ellipticam in puncto (uvw) tangente demissum, ex analytica geometria est

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}=\mp\frac{w}{cP};$$

ergo substitutione facta:

$$X-u=\pm\frac{u}{a^2}\cdot\frac{c}{w}\cdot PR,$$

$$Y-v=\pm\frac{v}{b^2}\cdot\frac{c}{w}\cdot PR,$$

$$Z-w=\frac{1}{w}\cdot PR;$$

et

$$\frac{R\frac{u}{a^2}}{X-u}=\frac{R\frac{v}{b^2}}{Y-v}=\pm\frac{R\frac{1}{c}}{Z-w}=\pm\frac{w}{cP}.$$

Jam vero linea recta in plano paraboloidem ellipticam in puncto (uvw) tangente sita, cujus aequationes sunt:

$$\frac{x-u}{\cos\theta}=\frac{y-v}{\cos\omega}=\frac{z-w}{\cos\overline{\omega}},$$

situm sectionis normalis definiri si ponimus, ex theoria obtinemus generali:

$$X-u=-\frac{\frac{u}{a^2}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}},$$

$$Y-v=-\frac{\frac{v}{b^2}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}},$$

$$Z-w=\mp\frac{\frac{1}{c}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}};$$

$$R=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}}.$$

Alteri sectioni normali, quae altera linea recta in eodem plano sita, cujus aequationes sunt:

$$\frac{x-u}{\cos\theta_1}=\frac{y-v}{\cos\omega_1}=\frac{z-w}{\cos\overline{\omega}_1},$$

determinata sit, hae prodeunt formulae, in quibus significationem symbolorum cuique satis perspicuam fore puto:

$$X_1 - u = -\frac{\frac{u}{a^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2}},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2}},$$

$$Z_1 - w = \mp\frac{\frac{1}{c}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2}};$$

$$R_1 = \frac{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2}}.$$

Quae duae sectiones normales si inter se sunt perpendiculares atque brevitatis causa

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} = \tfrac{1}{2}\Omega,$$

$$\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} = \tfrac{1}{2}\Omega_1$$

ponitur, theoria docet generalis hanc inter angulos θ, ω, ϖ et $\theta_1, \omega_1, \varpi_1$ esse aequationem:

$$\tfrac{1}{2}\Omega_1 = \frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} + \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}},$$

unde, quum sit (vid. Grunerti Arch. XXVIII. pag. 178.)

$$\left(\frac{\sin\theta^2}{a^2} + \frac{\sin\omega^2}{b^2}\right)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\} - \left\{\frac{u^2}{a^6} + \frac{v^2}{b^6}\right\}$$

$$= \frac{1}{a^2}\left\{\frac{v}{b^2}\cos\varpi \mp \frac{1}{c}\cos\omega\right\}^2 + \frac{1}{b^2}\left\{\pm\frac{1}{c}\cos\theta - \frac{u}{a^2}\cos\varpi\right\}^2,$$

adhibentes aequationem

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \pm \frac{2w}{c} = 0$$

facili calculo invenimus:

$$\tfrac{1}{2}\Omega_1 = \frac{a^2\cos\theta^2 + b^2\cos\omega^2 \mp 2c\cos\varpi\{u\cos\theta + v\cos\omega + w\cos\varpi\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}.$$

Ergo ad quantitates X_1, Y_1, Z_1; R_1 in hoc casu definiendas habemus:

$$X_1 - u = -\frac{\frac{u}{a^2}}{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}$$

$$= -\frac{ub^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{a^2\cos\theta^2 + b^2\cos\omega^2 \mp 2c\cos\varpi\{u\cos\theta + v\cos\omega + w\cos\varpi\}},$$

$$Y_1 - v = -\frac{\frac{v}{b^2}}{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}$$

$$= -\frac{vc^2a^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{a^2\cos\theta^2 + b^2\cos\omega^2 \mp 2c\cos\varpi\{u\cos\theta + v\cos\omega + w\cos\varpi\}},$$

$$Z_1 - w = \mp\frac{\frac{1}{c}}{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}$$

$$= \mp\frac{a^2b^2c\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{a^2\cos\theta^2 + b^2\cos\omega^2 \mp 2c\cos\varpi\{u\cos\theta + v\cos\omega + w\cos\varpi\}};$$

$$R_1 = \frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}$$

$$= \frac{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}{a^2\cos\theta^2 + b^2\cos\omega^2 \mp 2c\cos\varpi\{u\cos\theta + v\cos\omega + w\cos\varpi\}}.$$

Ex formulis superioribus inversi radii osculatorii duarum sectionum normalium, quae in puncto (uvw) paraboloidis ellipticae inter se sunt directae, sequuntur:

$$\frac{1}{R}=\frac{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=\frac{\frac{\sin\theta^2}{a^2}+\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}.$$

Itaque est:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{\frac{1}{a^2}+\frac{1}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}$$

$$=\frac{\frac{1}{a^2}\cdot\frac{v^2}{b^4}+\frac{1}{b^2}\cdot\frac{u^2}{a^4}+\frac{1}{c^2}\left\{\frac{1}{a^2}+\frac{1}{b^2}\right\}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}=\frac{a^2+b^2+c^2\left\{\frac{u^2}{a^2}+\frac{v^2}{b^2}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

vel, quum sit

$$c^2\left\{\frac{u^2}{a^2}+\frac{v^2}{b^2}\right\}=\mp 2cw,$$

$$\frac{1}{R}+\frac{1}{R_1}=\frac{a^2+b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

unde summam radiorum osculatoriorum inversorum, qui ad binas per idem paraboloidis ellipticae punctum positas et inter se perpendiculares sectiones normales relati sunt, omni paraboloidis ellipticae puncto constantem prodire apparet.

Porro est:

$$\frac{1}{R}-\frac{1}{R_1}=\frac{\frac{\cos 2\theta}{a^2}+\frac{\cos 2\omega}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}.$$

Quia ex formulis supra datis sunt:

$$\frac{1}{X-u}+\frac{1}{X_1-u}=\pm\frac{a^2w}{cu}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\}$$

$$=-\tfrac{1}{2}\cdot\frac{a^2v^2+b^2u^2}{b^2u}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\},$$

$$\frac{1}{Y-v}+\frac{1}{Y_1-v}=\pm\frac{b^2w}{cv}\cdot\frac{1}{P}\left\{\frac{1}{R}+\frac{1}{R_1}\right\}$$

$$=-\tfrac{1}{2}\cdot\frac{a^2v^2+b^2u^2}{a^2v}\cdot\frac{1}{P}\cdot\left\{\frac{1}{R}+\frac{1}{R_1}\right\},$$

$$\frac{1}{Z-w}+\frac{1}{Z_1-w}=c\cdot\frac{1}{P}\cdot\left\{\frac{1}{R}+\frac{1}{R_1}\right\};$$

in omni paraboloidis ellipticae puncto binis sectionibus normalibus, quae inter se sunt perpendiculares summas quoque

$$\frac{1}{X-u}+\frac{1}{X_1-u},\quad \frac{1}{Y-v}+\frac{1}{Y_1-v},\quad \frac{1}{Z-w}+\frac{1}{Z_1-w}$$

constantes esse apparet.

III.

Sectionibus normalibus, quae ad curvaturam maximum et minimum praebeant, abhinc occupabimur.

Sectiones eas brevitatis causa vocabimus principales.

Sectiones duas normales et inter se directas, quas modo tractavimus, principales esse paraboloidis ellipticae si ponimus, ex theoria generali inter Ω et Ω_1 has duas obtinemus aeqnationes:

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{1}{a^2}+\frac{1}{b^2}-\frac{\frac{u^2}{a^6}+\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}$$

vel, quod jam supra demonstravimus,

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{a^2+b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

$$\tfrac{1}{4}\Omega\Omega_1=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{-1};$$

ex quibus facili calculo invenimus:

$$\Omega = \frac{\left\{ \begin{array}{l} a^2+b^2\mp 2cw \\ (\pm)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}} \end{array} \right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

$$\Omega_1 = \frac{\left\{ \begin{array}{l} a^2+b^2\mp 2cw \\ (\mp)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}} \end{array} \right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

ubi signa (uncis inclusa) superiora ad superiora, inferiora ad inferiora referenda sunt.

E §. II sunt:

$$\frac{1}{R} = \frac{\frac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}$$

et

$$\frac{1}{R_1} = \frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ergo:

$$\frac{1}{R}+\frac{1}{R_1} = \frac{\frac{1}{2}(\Omega+\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{RR_1} = \frac{\frac{1}{4}\Omega\Omega_1}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}},$$

$$R+R_1 = \frac{\frac{1}{2}(\Omega+\Omega_1)}{\frac{1}{4}\Omega\Omega_1}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}};$$

vel, valoribus pro $\frac{1}{2}(\Omega+\Omega_1)$ et $\frac{1}{4}\Omega\Omega_1$ substitutis:

$$\frac{1}{R}+\frac{1}{R_1} = \frac{a^2+b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

$$R+R_1 = \{a^2+b^2\mp 2cw\}\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}},$$

$$\frac{1}{RR_1} = \frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{-2};$$

vel perpendiculo P substituto:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{P^3}{w^3}\cdot c\cdot\frac{\mp(a^2+b^2)+2cw}{a^2b^2},$$

$$R+R_1=\frac{w}{c}\cdot\frac{\mp(a^2+b^2)+2cw}{P},$$

$$\frac{1}{RR_1}=\frac{c^2}{a^2b^2}\cdot\frac{P^4}{w^4}$$

vel

$$RR_1=\frac{a^2b^2}{c^2}\cdot\frac{w^4}{P^4}$$

vel

$$\frac{a^2b^2}{c^2}=\frac{P^4}{w^4}\cdot RR_1$$

vel etiam:

$$P=\mp w\sqrt[4]{\frac{a^2b^2:c^2}{RR_1}}.$$

Formula

$$RR_1=a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^2$$

divisa per

$$R=\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}},$$

pro radio curvaturae R_1 expressionem

$$R_1=a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}\cdot\left\{\frac{\cos\theta^2}{a^2}+\frac{\cos\omega^2}{b^2}\right\}$$

effici, memoratu dignum nobis videtur esse.

Jam vero in formulis

$$\frac{1}{R}=\frac{\frac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=\frac{\frac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

si loco Ω et Ω_1 valores eorum substituimus, pro radiis sectionum principalium osculatoriis expressiones obtinemus memorabiles hasce:

$$\frac{2}{R}=\frac{\left\{a^2+b^2\mp 2cw\,(\pm)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

$$\frac{2}{R_1}=\frac{\left\{a^2+b^2\mp 2cw\,(\mp)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

ubi signa (uncis inclusa) superiora et inferiora in se invicem referenda sunt.

Sectiones principales ad quantitates Ω et Ω_1 maximum alteram, alteram minimum praebere theoria docet generalis.

Conditiones, quibus Ω et Ω_1 maxima ac minima sunt, nunc constituemus.

E theoria generali sunt:

$$\frac{d^2\Omega}{d\,\theta^2}=2\sin\theta^2\,.\,\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}{\left\{\frac{v}{b^2}\cos\overline{\omega}\mp\frac{1}{c}\cos\omega\right\}^2}\,.\,(\Omega_1-\Omega),$$

$$\frac{d^2\Omega_1}{d\theta_1{}^2}=2\sin\theta_1{}^2\,.\,\frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}{\left\{\frac{v}{b^2}\cos\overline{\omega}_1\mp\frac{1}{c}\cos\omega_1\right\}^2}\,.\,(\Omega-\Omega_1),$$

unde ex principiis calculi differentialis concludimus:

si $\Omega-\Omega_1>0$ est, Ω maximum et Ω_1 minimum est; sin $\Omega-\Omega_1<0$ est, Ω minimum et Ω_1 maximum est.

In iis, quae sequuntur, situs sectionum principalium in universum definietur.

Quo consilio quantitates M, M', M'' in theoriam generalem brevitatis causa inductae antea determinentur necesse est.

Eae facile prodeunt:

$$M=-\frac{2u\Omega\,(2-b^2\,\Omega)}{a^2b^2},$$

$$M'=-\frac{2v\Omega\,(2-a^2\,\Omega)}{a^2b^2},$$

$$M'' = \pm \frac{2.(2-a^2\,\Omega)(2-b^2\,\Omega)}{a^2b^2c};$$

ergo:

$$M^2 + M'^2 + M''^2$$

$$= 4 . \frac{(2-a^2\Omega)^2(2-b^2\Omega)^2}{a^4b^4} . \left\{ \left(\frac{u\Omega}{2-a^2\,\Omega}\right)^2 + \left(\frac{v\Omega}{2-b^2\Omega}\right)^2 + \frac{1}{c^2} \right\}.$$

Ex theoria igitur generali ad determinandos angulos θ, ω, $\overline{\omega}$, quibus situs sectionum principalium definitur, formulas obtinemus hasce:

$$\cos\theta = (\pm) \frac{\dfrac{u\Omega}{2-a^2\,\Omega}}{\left\{\left(\dfrac{u\Omega}{2-a^2\,\Omega}\right)^2 + \left(\dfrac{v\,\Omega}{2-b^2\,\Omega}\right)^2 + \dfrac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\cos\omega = (\pm) \frac{\dfrac{v\,\Omega}{2-b^2\,\Omega}}{\left\{\left(\dfrac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\dfrac{v\,\Omega}{2-b^2\,\Omega}\right)^2 + \dfrac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\cos\overline{\omega} = (\pm) \mp \frac{\dfrac{1}{c}}{\left\{\left(\dfrac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\dfrac{v\Omega}{2-b^2\Omega}\right)^2 + \dfrac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ubi signa superiora et inferiora in se invicem referenda sunt, sed utra sumantur, superiora an inferiora, nihil interest, quum iis una modo et eadem linea recta per punctum (uvw) ducta determinetur. Formulis vero illis propter duplicem quantitatis Ω valorem duos eosque, ut hoc modo effici potest, reales pro angulis reddi valores, statim apparet.

Valores, quos pro $\cos\theta$, $\cos\omega$, $\cos\overline{\omega}$ invenimus, si Ω est reale, reales prodire, statim perspicitur. Ω vero quantitas realis est, si est

$$\{a^2 + b^2 \mp 2wc\}^2 - 4\,a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\} \geqq 0.$$

Quae differentia, si ponimus

$$\mp 2wc = c^2 \left\{\frac{u^2}{a^2} + \frac{v^2}{b^2}\right\},$$

abit in

$$\left\{a^2 + b^2 + c^2\left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right)\right\}^2 - 4a^2b^2c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}$$

et inde in

$$(a^2+b^2)^2+c^4\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)^2+2c^2(a^2+b^2)\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}.$$

Sunt vero:

$$(a^2+b^2)^2-4a^2b^2=(a^2-b^2)^2,$$

$$(a^2-b^2)^2+c^4\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)^2=\left\{a^2-b^2+c^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)\right\}^2-2c^2(a^2-b^2)\left\{\frac{u^2}{a^2}+\frac{v^2}{b^2}\right\}$$

$$=\left\{a^2-b^2-c^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)\right\}^2+2c^2(a^2-b^2)\left\{\frac{u^2}{a^2}+\frac{v^2}{b^2}\right\},$$

$$2c^2(a^2+b^2)\left\{\frac{u^2}{a^2}+\frac{v^2}{b^2}\right\}-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}\right\}=2c^2(a^2-b^2)\left\{\frac{u^2}{a^2}-\frac{v^2}{b^2}\right\};$$

ergo substitutione facta pro differentia illa obtinemus expressionem hanc:

$$\left\{a^2-b^2+c^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)\right\}^2-4c^2(a^2-b^2)\frac{v^2}{b^2},$$

vel hanc:

$$\left\{a^2-b^2-c^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)\right\}^2+4c^2(a^2-b^2)\frac{u^2}{a^2},$$

quibus differentiam, de qua quaerimus, in negativum abire valorem nunquam posse efficitur.

Coordinatae denique centrorum curvaturarum sectionum principalium ex theoria generali determinantur formulis hisce:

$$X-u=-2.\frac{u\,b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\left\{\begin{matrix}a^2+b^2\mp 2cw\\(\pm)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\end{matrix}\right\}},$$

$$Y-v=-2.\frac{v\,c^2a^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\left\{\begin{matrix}a^2+b^2\mp 2cw\\(\pm)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\end{matrix}\right\}},$$

$$Z-w=\mp 2.\frac{a^2b^2c\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\left\{\begin{matrix}a^2+b^2\mp 2cw\\(\pm)\sqrt{\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\end{matrix}\right\}},$$

ubi signa (uncis inclusa) superiora uni sectioni, alteri inferiora valent.

Ad ultimum nobis liceat quaerere, quos valores et anguli, quibus sectionum principalium situs determinatur, et radii sectionum earum osculatorii induant, si punctum (uvw) in intersectione plani cujusvis coordinatarum et paraboloidis ellipticae positum est.

Hoc in casu, quum longior postuletur calculus atque molestior, si ex formulis supra datis quam si ex aequationibus, quibus in universum anguli illi paraboloidi ellipticae determinantur, angulos computamus, ad eos determinandos viam posteriorem ingredimur.

Aequationes illae sunt:

$$(1) \qquad \frac{u}{a^2}\cos\theta + \frac{v}{b^2}\cos\omega \pm \frac{1}{c}\cos\varpi = 0,$$

$$(2) \qquad \pm\frac{1}{c}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\cos\theta\cos\omega + \frac{1}{b^2}\cdot\frac{u}{a^2}\cos\omega\cos\varpi - \frac{1}{a^2}\cdot\frac{v}{b^2}\cos\varpi\cos\theta = 0,$$

$$(3) \qquad \cos\theta^2 + \cos\omega^2 + \cos\varpi^2 = 1.$$

In quibus primum si ponimus

$$u = 0,$$

aequationi (2) satisfit, si est

$$\cos\theta = 0$$

aut

$$\pm(a^2 - b^2)\cos\omega + cv\cos\varpi = 0.$$

Si est

$$\cos\theta = 0,$$

ex aequat. (1) et (3) efficitur:

$$\cos\omega = (\pm)\frac{b^2}{\sqrt{b^4 + c^2v^2}},$$

$$\cos\varpi = (\pm)\mp\frac{cv}{\sqrt{b^4 + c^2v^2}};$$

ergo, quum sit

$$\frac{\cos\theta^2}{a^2} + \frac{\cos\omega^2}{b^2} = \frac{b^2}{b^4 + c^2v^2},$$

$$R = \frac{\{b^4 + c^2v^2\}^{\frac{3}{2}}}{b^4c};$$

sin vero est

(2) $$\pm(a^2 - b^2)\cos\omega + cv\cos\overline{\omega} = 0$$

et valores, in quos hoc casu anguli θ, ω, $\overline{\omega}$ et radius curvaturae R abeunt, litteris θ_1, ω_1, $\overline{\omega}_1$; R_1 significantur, ex aequat. illa et (1) facili calculo sequitur:

$$\{(a^2 - b^2)b^2 - c^2v^2\}\cos\omega_1 = 0,$$

unde

$$\cos\omega_1 = 0,$$

ideoque ex aeq. (1)

$$\cos\overline{\omega}_1 = 0,$$

et ex (3)

$$\cos\theta_1 = \pm 1;$$

ergo, quia

$$\frac{\cos\theta_1{}^2}{a^2} + \frac{\cos\omega_1{}^2}{b^2} = \frac{1}{a^2},$$

$$R_1 = a^2 . \frac{\{b^4 + c^2v^2\}^{\frac{1}{2}}}{b^2c}.$$

Deinde si ponimus

$$v = 0,$$

eodem facili calculo obtinemus:

$$\cos\theta = (\pm)\frac{a^2}{\sqrt{a^4 + c^2u^2}}, \quad \cos\omega = 0, \quad \cos\overline{\omega} = (\pm)\mp\frac{cu}{\sqrt{a^4 + c^2u^2}},$$

$$R = \frac{\{a^4 + c^2u^2\}^{\frac{3}{2}}}{a^4c};$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = 0, \quad \cos\overline{\omega}_1 = (\pm)1,$$

$$R_1 = b^2 . \frac{\{a^4 + c^2u^2\}^{\frac{1}{2}}}{a^2c};$$

tum si ponimus

$$w = 0,$$

$$\cos\theta = (\pm)1, \quad \cos\omega = 0, \quad \cos\overline{\omega} = 0,$$

$$R = \frac{a^2}{c};$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = (\pm)1, \quad \cos\overline{\omega}_1 = 0,$$

$$R_1 = \frac{b^2}{c};$$

quo in calculo non est negligendum, si ponimus $w=0$, ex aequatione paraboloidis ellipticae $u=0$ et $v=0$ sit necesse esse.

Porro quum aequatio plani paraboloidem ellipticam in puncto $(u=0,\ v=0,\ w=0)$ tangentis sit

$$z=0,$$

ergo planum tangens cum plano (xy) ideoque normalis in puncto illo erecta cum z axe coincidat, facili argumentatione efficitur:

Si punctum (uvw) in intersectione planorum (xz), (yz) et paraboloidis ellipticae positum est, altera linea, quae sita in plano paraboloidem ellipticam in puncto (uvw) tangente alteram sectionem principalem paraboloidis ellipticae in puncto eo determinat, ei semper axi coordinatarum parallela est, qui ad planum, in quo punctum jacet, est perpendicularis; altera vero linea, quae in eodem plano sita alteram sectionem principalem in eo puncto paraboloidis ellipticae determinat, ad axem illum directa est; sin punctum (uvw) in initium incidit coordinatarum systematis, sectiones, quae planis coordinatarum paraboloidem ellipticam persecantibus efficiuntur, sunt principales.

Paraboloides elliptica rotatione descripta.

In generali paraboloidis ellipticae aequatione si ponimus $a=b$, aequationem obtinemus paraboloidis, quae rotatione parabolae circa z axem descripta est.

Si vero est $a=b$, quum expressione, in quam supra differentiam

$$\{a^2+b^2\mp 2cw\}^2-4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}$$

transmutavimus, differentiam appareat eam abire in

$$\left\{c^2\,.\,\frac{u^2+v^2}{a^2}\right\}^2,$$

Ω quantitas, ratione habita aequationis

$$c^2\,.\,\frac{u^2+v^2}{a^2}=\mp 2cw,$$

signo si utimur superiore, facili calculo prodit:

$$\Omega = \frac{2}{a^2};$$

sin inferiore:

$$\Omega = \frac{2}{a^2 c^2 \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}}.$$

In priori igitur casu, quum sit

$$2 - a^2 \Omega = 0,$$

cosinus angulorum θ, ω, $\overline{\omega}$ prodeunt hi:

$$\cos\theta = \frac{\infty}{\infty}, \quad \cos\omega = \frac{\infty}{\infty}, \quad \cos\overline{\omega} = 0;$$

in altero, quum sit

$$2 - a^2 \Omega = \frac{u^2 + v^2}{a^4 \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}},$$

cosinus angulorum θ_1, ω_1, $\overline{\omega}_1$, quibus alterius sectionis principalis situs determinetur, facili calculo prodeunt hi:

$$\cos\theta_1 = (\pm) \frac{a^2 u}{\sqrt{(u^2 + v^2)\{a^4 + (u^2 + v^2) c^2\}}},$$

$$\cos\omega_1 = (\pm) \frac{a^2 v}{\sqrt{(u^2 + v^2)\{a^4 + (u^2 + v^2) c^2\}}},$$

$$\cos\overline{\omega}_1 = (\pm)\, 2 . \frac{a^2 w}{\sqrt{(u^2 + v^2)\{a^4 + (u^2 + v^2) c^2\}}};$$

vel

$$\cos\theta_1 = (\pm) \frac{u}{c \sqrt{(u^2 + v^2) \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}}},$$

$$\cos\omega_1 = (\pm) \frac{v}{c \sqrt{(u^2 + v^2) \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}}},$$

$$\cos\overline{\omega}_1 = (\pm)\, 2 . \frac{w}{c \sqrt{(u^2 + v^2) \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}}};$$

vel etiam, quum facillime inveniatur,

$$c \sqrt{(u^2 + v^2) . \left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}} = \sqrt{u^2 + v^2 + 4w^2}:$$

$$\cos\theta_1 = (\pm) \frac{u}{\sqrt{u^2 + v^2 + 4w^2}},$$

$$\cos \omega_1 = (\pm) \frac{v}{\sqrt{u^2 + v^2 + 4w^2}},$$

$$\cos \overline{\omega}_1 = (\pm) 2 . \frac{w}{\sqrt{u^2 + v^2 + 4w^2}}.$$

In priori casu quum in valoribus angulorum θ et ω computandis in medio relicti simus, ut angulorum eorum valores cognoscamus, ex aequationibus, quibus anguli illi paraboloidi rotatione descriptae definiuntur, valores, in quos cosinus angulorum (θ et ω) abeunt, si est $\cos \overline{\omega} = 0$, computabimus.

Aequationes illae sunt:

(1) $$\frac{u}{a^2} \cos \theta + \frac{v}{a^2} \cos \omega \pm \frac{1}{c} \cos \overline{\omega} = 0,$$

(2) $$\{u \cos w - v \cos \theta\} \cos \overline{\omega} = 0,$$

(3) $$\cos \theta^2 + \cos \omega^2 + \cos \overline{\omega}^2 = 1;$$

ex quibus, $\cos \overline{\omega} = 0$ posito, facili calculo sequuntur:

$$\cos \theta = (\pm) \frac{v}{\sqrt{u^2 + v^2}},$$

$$\cos \omega = (\mp) \frac{u}{\sqrt{u^2 + v^2}}.$$

In omnibus hisce formulis signa (uncis inclusa) superiora et inferiora in se invicem referenda sunt, sed utra sumantur, superiora an inferiora, nihil interest, quum iis una modo linea recta determinetur.

Ex formulis superioribus obtinemus:

$$\frac{\cos \theta^2 + \cos \omega^2}{a^2} = \frac{1}{a^2}$$

et

$$\frac{\cos \theta_1{}^2 + \cos \omega_1{}^2}{a^2} = \frac{a^2}{a^4 + (u^2 + v^2) c^2};$$

ergo, quum sint

$$R = \frac{\left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}^{\frac{1}{2}}}{\frac{\cos \theta^2 + \cos \omega^2}{a^2}},$$

$$R_1 = \frac{\left\{ \frac{u^2 + v^2}{a^4} + \frac{1}{c^2} \right\}^{\frac{1}{2}}}{\frac{\cos \theta_1{}^2 + \cos \omega_1{}^2}{a^2}};$$

substitutione facta:

$$R = a^2 \cdot \left\{\frac{u^2+v^2}{a^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}} = \frac{\{a^4+(u^2+v^2)c^2\}^{\frac{1}{2}}}{c},$$

$$R_1 = a^2c^2 \left\{\frac{u^2+v^2}{a^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}} = \frac{\{a^4+(u^2+v^2)c^2\}^{\frac{3}{2}}}{a^4 c};$$

unde:

$$RR_1 = a^4c^2 \left\{\frac{u^2+v^2}{a^4} + \frac{1}{c^2}\right\}^2 = \frac{\{a^4+(u^2+v^2)c^2\}^2}{a^4c^2} = \frac{\{a^2 \mp 2cw\}^2}{c^2},$$

$$R + R_1 = 2.(a^2 \mp cw)\left\{\frac{u^2+v^2}{a^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}},$$

$$\frac{1}{R} + \frac{1}{R_1} = 2 \cdot \frac{a^2 \mp cw}{a^4c^2\left\{\frac{u^2+v^2}{a^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}}}.$$

Paraboloides hyperbolica.

I.

Paraboloidis hyperbolicae aequatio sit:

$$u = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \pm \frac{2z}{c} = 0. \text{ *)}$$

In iis, quae sequuntur, aequationibus ac formulis signa duplicia, quae ad signa duplicia in aequatione paraboloidis hyperbolicae non sunt referenda, ut errores effugiamus, uncis includantur. Signa vero duplicia, quae uncis inclusa non sunt, superiora ad superius, inferiora ad inferius in aequatione paraboloidis hyperbolicae signum referenda sunt.

Functione u partialiter ad x, y, z differentiata primae et secundae partiales quotientes differentiales functionis u efficiuntur:

$$\frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = -\frac{2y}{b^2}, \quad \frac{du}{dz} = \pm\frac{2}{c};$$

$$\frac{d^2u}{dx^2} = \frac{2}{a^2}, \quad \frac{d^2u}{dy^2} = -\frac{2}{b^2}, \quad \frac{d^2u}{dz^2} = 0;$$

$$\frac{d^2u}{dxdy} = 0, \quad \frac{d^2u}{dydz} = 0, \quad \frac{d^2u}{dzdx} = 0.$$

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*) De paraboloide hyperb. vid. Grunerti geom. analyt. part. II. §. 80. pag. 218 sq.

Si porro coordinatas puncti cujusvis in superficie paraboloidis hyperbolicae siti per (uvw), variabiles vero per (xyz) significamus, e theoria generali aequationem plani paraboloidem hyperbolicam in puncto (uvw) tangentis obtinemus hanc:

$$\frac{u}{a^2}(x-u)-\frac{v}{b^2}(y-v)\pm\frac{1}{c}(z-w)=0$$

vel, quum sit

$$\left(\frac{u}{a}\right)^2-\left(\frac{v}{b}\right)^2\pm\frac{2w}{c}=0,$$

hanc:

$$\frac{ux}{a^2}-\frac{vy}{b^2}\pm\frac{1}{c}z=\mp\frac{w}{c}.$$

Aequatio plani cujusvis per punctum (uvw) paraboloidis hyperbolicae positi sit:

$$A(x-u)+B(y-v)+C(z-w)=0$$

et curva, quae plano eo paraboloidem hyperbolicam persecante efficitur, brevitatis causa sectio vocetur.

Quam sectionem nunc tractabimus.

Theoria generali aequationes lineae sectionem in puncto (uvw) tangentis invenimus has:

$$\frac{x-u}{\pm B\frac{1}{c}+C\frac{v}{b^2}}=\frac{y-v}{C\frac{u}{a^2}\mp A\frac{1}{c}}=-\frac{z-w}{A\frac{v}{b^2}+B\frac{u}{a^2}};$$

hanc plani normalis:

$$\left\{\pm B\frac{1}{c}+C\frac{v}{b^2}\right\}(x-u)+\left\{C\frac{u}{a^2}\mp A\frac{1}{c}\right\}(y-v)-\left\{A\frac{v}{b^2}+B\frac{u}{a^2}\right\}(z-w)=0;$$

has normalis in illo sectionis puncto erectae:

$$\frac{x-u}{A\left\{A\frac{u}{a^2}-B\frac{v}{b^2}\pm C\frac{1}{c}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2}}$$

$$=\frac{y-v}{B\left\{A\frac{u}{a^2}-B\frac{v}{b^2}\pm C\frac{1}{c}\right\}+\left\{A^2+B^2+C^2\right\}\frac{v}{b^2}}$$

$$=\frac{z-w}{C\left\{A\frac{u}{a^2}-B\frac{v}{b^2}\pm C\frac{1}{c}\right\}\mp\left\{A^2+B^2+C^2\right\}\frac{1}{c}}.$$

Porro quum quantitates in theoriam universalem brevitatis causa inductae exsistant:

$$\tfrac{1}{2}U=A\left\{A\frac{u}{a^2}-B\frac{v}{b^2}\pm C\frac{1}{c}\right\}-\left\{A^2+B^2+C^2\right\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V = B\{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\} + \{A^2 + B^2 + C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = C\{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\} \mp \{A^2 + B^2 + C^2\}\frac{1}{c};$$

$$\tfrac{1}{4}Q^2 = \{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2 + \{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 + \{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2$$

$$= \{A^2 + B^2 + C^2\}\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\} - \{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\}^2,$$

$$\tfrac{1}{8}\Theta = \frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2;$$

ad coordinatas centri (XYZ) ac radium R curvaturae in puncto (uvw) sectionis determinanda habemus formulas has:

$$\frac{X-u}{A\{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\} - \{A^2 + B^2 + C^2\}\frac{u}{a^2}}$$

$$= \frac{Y-v}{B\{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\} + \{A^2 + B^2 + C^2\}\frac{v}{b^2}}$$

$$= \frac{Z-w}{C\{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\} \mp \{A^2 + B^2 + C^2\}\frac{1}{c}}$$

$$= \frac{\{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2 + \{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 + \{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]}$$

$$= \frac{(A^2 + B^2 + C^2)\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\} - \{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\}^2}{(A^2+B^2+C^2)\left[\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]}$$

atque

$$R = (\pm)\frac{[\{A\frac{v}{b^2} + B\frac{u}{a^2}\}^2 + \{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 + \{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\cdot\left[\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]}$$

$$= (\pm)\frac{[(A^2 + B^2 + C^2)\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\} - \{A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c}\}^2]^{\frac{3}{2}}}{\sqrt{A^2+B^2+C^2}\left[\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2\right]},$$

ubi signa sumenda sunt ita, ut R sit affirmativum.

Adhibentes aequationem

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 \pm \frac{2w}{c} = 0$$

expressionem pro $\frac{1}{8}\Theta$ supra datam facili calculo transfiguramus in hanc:

$$-\frac{A^2a^2 - B^2b^2 \mp 2Cc\{Au + Bv + Cw\}}{a^2b^2c^2},$$

quae in formulis substitui potest, si quando usui est.

II.

Sectiones normales.

Planum aequatione

$$A(x-u) + B(y-v) + C(z-w) = 0$$

definitum si per normalem in puncto (uvw) paraboloidis hyperbolicae erectam transire ponimus, ex analytica geometria sit necesse est

$$A\frac{u}{a^2} - B\frac{v}{b^2} \pm C\frac{1}{c} = 0.$$

Sunt igitur in hoc casu:

$$\tfrac{1}{2}U = -\{A^2 + B^2 + C^2\}\frac{u}{a^2},$$

$$\tfrac{1}{2}V = \{A^2 + B^2 + C^2\}\frac{v}{b^2},$$

$$\tfrac{1}{2}W = \mp\{A^2 + B^2 + C^2\}\frac{1}{c};$$

$$\tfrac{1}{4}Q^2 = (A^2 + B^2 + C^2)\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\};$$

ergo:

$$X - u = -\frac{u}{a^2}\cdot\frac{\{A^2 + B^2 + C^2\}\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2},$$

$$Y - v = \frac{v}{b^2}\cdot\frac{\{A^2 + B^2 + C^2\}\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c} + C\frac{v}{b^2}\}^2 - \frac{1}{b^2}\{C\frac{u}{a^2} \mp A\frac{1}{c}\}^2},$$

$$Z-w=\mp\frac{1}{c}\cdot\frac{\{A^2+B^2+C^2\}\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{\frac{1}{a^2}\{\pm B\frac{1}{c}+C\frac{v}{b^2}\}^2-\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2};$$

$$R=(\pm)\frac{\{A^2+B^2+C^2\}\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}{\frac{1}{a^2}\{\pm B\frac{1}{c}+C\frac{v}{b^2}\}^2-\frac{1}{b^2}\{C\frac{u}{a^2}\mp A\frac{1}{c}\}^2};$$

ubi signum sumendum est ita, ut R fiat affirmativum.

Quibus ex formulis consequuntur:

$$X-u=(\mp)\frac{u}{a^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$Y-v=(\pm)\frac{v}{b^2}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$Z-w=(\pm)\mp\frac{1}{c}\cdot\frac{R}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}};$$

unde:

$$\frac{R\frac{u}{a^2}}{X-u}=-\frac{R\frac{v}{b^2}}{Y-v}=\pm\frac{R\frac{1}{c}}{Z-w}=(\mp)\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}.$$

ubi signa (uncis inclusa) superiora aut inferiora patet esse sumenda, prout $\frac{1}{8}\Theta$ affirmativum aut negativum prodit.

Porro si perpendiculum ab initio coordinatarum systematis in plano paraboloidem hyperbolicam in puncto (uvw) tangente demissum per P denotamus, ex analyticae geometriae principiis est

$$\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}=\frac{w^2}{c^2}\cdot\frac{1}{P^2},$$

quod in formulis superioribus substitui potest.

Per punctum (uvw) paraboloidis hyperbolicae lineam quandam rectam, cujus aequationes sint:

$$\frac{x-u}{\cos\theta}=\frac{y-v}{\cos\omega}=\frac{z-w}{\cos\overline{\omega}},$$

si ductam cogitamus atque linea ea ac normali in puncto (uvw) paraboloidis hyperbolicae erecta situm plani sectionis definitum esse ponimus, theoria generali obtinemus formulas has:

$$X-u=-\frac{\frac{u}{a^2}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}},$$

$$Y-v=\frac{\frac{v}{b^2}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}},$$

$$Z-w=\mp\frac{\frac{1}{c}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}};$$

$$R=(\pm)\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}},$$

ubi signum sumendum est ita, ut R fiat positivum.

Alteri sectioni normali, quae eodem, ut praecedens, modo sed linea recta, cujus aequationes sunt

$$\frac{x-u}{\cos\theta_1}=\frac{y-v}{\cos\omega_1}=\frac{z-w}{\cos\overline{\omega}_1},$$

determinata sit, formulae prodeunt hae, in quibus symbolorum significationem satis perspicuam fore puto:

$$X_1-u=-\frac{\frac{u}{a^2}}{\frac{\cos{\theta_1}^2}{a^2}-\frac{\cos{\omega_1}^2}{b^2}},$$

$$Y_1-v=\frac{\frac{v}{b^2}}{\frac{\cos{\theta_1}^2}{a^2}-\frac{\cos{\omega_1}^2}{b^2}},$$

$$Z_1-w=\mp\frac{\frac{1}{c}}{\frac{\cos{\theta_1}^2}{a^2}-\frac{\cos{\omega_1}^2}{b^2}};$$

$$R_1=(\pm)\frac{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}{\frac{\cos{\theta_1}^2}{a^2}-\frac{\cos{\omega_1}^2}{b^2}},$$

ubi signum sumendum est ita, ut R fiat affirmativum.

Quae duae sectiones normales si ad se sunt perpendiculares ac brevitatis causa ponitur

$$\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}=\tfrac{1}{2}\Omega\,,$$

$$\frac{\cos\theta_1{}^2}{a^2}-\frac{\cos\omega_1{}^2}{b^2}=\tfrac{1}{2}\Omega_1\,,$$

theoria generali inter angulos θ, ω, $\overline{\omega}$ et θ_1, ω_1, $\overline{\omega}_1$ habemus aequationem:

$$\tfrac{1}{2}\Omega_1=\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}\,,$$

quam, quum sit (vid. Gruner ti Arch. XXVIII. pag. 178.)

$$\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}\left\{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}\right\}-\left\{\frac{u^2}{a^6}-\frac{v^2}{b^6}\right\}$$
$$=\frac{1}{a^2}\cdot\left\{\frac{v}{b^2}\cos\overline{\omega}\pm\frac{1}{c}\cos\omega\right\}^2-\frac{1}{b^2}\left\{\pm\frac{1}{c}\cos\theta-\frac{u}{a^2}\cos\overline{\omega}\right\}^2,$$

ratione habita aequationis

$$\left(\frac{u}{a}\right)^2-\left(\frac{v}{b}\right)^2\pm\frac{2w}{c}=0$$

facili calculo transmutamus in

$$\tfrac{1}{2}\Omega_1=-\frac{a^2\cos\theta^2-b^2\cos\omega^2\mp 2c\cos\overline{\omega}\,\{u\cos\theta+v\cos\omega+w\cos\overline{\omega}\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}.$$

Quare ad quantitates X_1, Y_1, Z_1; R_1 definiendas obtinemus formulas, quibus quantitates illae ab angulis θ, ω, $\overline{\omega}$ dependentes datae sunt:

$$X_1-u=-\frac{\frac{u}{a^2}}{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}$$

$$=\frac{ub^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}{a^2\cos\theta^2-b^2\cos\omega^2\mp 2c\cos\overline{\omega}\,\{u\cos\theta+v\cos\omega+w\cos\overline{\omega}\}},$$

$$Y_1 - v = \frac{\frac{v}{b^2}}{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}}}$$

$$= - \frac{vc^2 a^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}}{a^2 \cos\theta^2 - b^2 \cos\omega^2 \mp 2c\cos\overline{\omega} \{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}},$$

$$Z_1 - w = \mp \frac{\frac{1}{c}}{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}}}$$

$$= \pm \frac{a^2 b^2 c \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}}{a^2 \cos\theta^2 - b^2 \cos\omega^2 \mp 2c\cos\overline{\omega} \{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}};$$

$$R_1 = (\pm) \frac{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}^{\frac{1}{2}}}{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}}}$$

$$= (\pm) \frac{a^2 b^2 c^2 \left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}^{\frac{3}{2}}}{a^2 \cos\theta^2 - b^2 \cos\omega^2 \mp 2c\cos\overline{\omega} \{u\cos\theta + v\cos\omega + w\cos\overline{\omega}\}},$$

ubi signa sumenda sunt ita, ut R_1 fiat affirmativum.

Ex formulis superioribus inversi radii osculatorii duarum sectionum normalium, quae in puncto (uvw) paraboloidis hyperbolicae inter se ad perpendiculum directae sunt, sequuntur:

$$\frac{1}{R} = (\pm) \frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1} = (\pm) \frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}}}{\left\{ \frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2} \right\}^{\frac{1}{2}}},$$

ubi signa superiora aut inferiora sumenda sunt, prout fractionum numeratores, id est prout quantitates Ω et Ω_1 positivae aut negativae sunt.

Quaerentibus, utrum quantitates illae iisdem an diversis semper praeditae sint signis, statim apparebit ea de re in universum nihil pro certo constitui posse.

Signa vero quantitatum earum quod attinet, duo modo sunt casus, qui per rerum naturam admitti possunt; iisdem enim aut diversis signis quantitates illae praeditae sint necesse est.

Primum quantitatibus eadem signa esse si ponimus, formulas obtinemus has:

$$\frac{1}{R} = (\pm) \frac{\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}}}$$

$$\frac{1}{R_1} = (\pm) \frac{\frac{\sin\theta^2}{a^2} - \frac{\sin\omega^2}{b^2} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}}}.$$

ubi signa (uncis inclusa) in se invicem referenda sed superiora aut inferiora sumenda sunt, prout quantitates Ω et Ω_1 positivae aut negativae prodeunt.

Eadem signorum determinatione sunt:

$$\frac{1}{R} + \frac{1}{R_1} = (\pm)\left[\frac{\frac{1}{a^2} - \frac{1}{b^2}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{1}{2}}} - \frac{\frac{u^2}{a^6} - \frac{v^2}{b^6}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}}}\right]$$

$$= (\pm) \frac{\frac{1}{a^2}\cdot\frac{v^2}{b^4} - \frac{1}{b^2}\cdot\frac{u^2}{a^4} + \frac{1}{c^2}\left\{\frac{1}{a^2} - \frac{1}{b^2}\right\}}{\left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}}} = (\mp) \frac{a^2 - b^2 + c^2\left\{\frac{u^2}{a^2} - \frac{v^2}{b^2}\right\}}{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

vel, quum sit

$$c^2\left\{\frac{u^2}{a^2} - \frac{v^2}{b^2}\right\} = \mp 2cw,$$

$$\frac{1}{R} + \frac{1}{R_1} = (\mp) \frac{a^2 - b^2 \mp 2cw}{a^2 b^2 c^2 \left\{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

atque

$$\frac{1}{R}-\frac{1}{R_1}=(\pm)\left[\frac{\frac{\cos 2\theta}{a^2}-\frac{\cos 2\omega}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}\right].$$

Sin vero quantitates Ω et Ω_1 diversis signis praeditae sunt, obtinemus formulas has:

$$\frac{1}{R}=(\pm)\frac{\frac{\cos\theta^2}{a^2}-\frac{\cos\omega^2}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=(\mp)\frac{\frac{\sin\theta^2}{a^2}-\frac{\sin\omega^2}{b^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\frac{u^2}{b^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

ubi signa in se invicem referenda sed superiora aut inferiora sumenda sunt, prout Ω positivum aut negativum est.

Eadem signorum determinatione sunt:

$$\frac{1}{R}+\frac{1}{R_1}=(\pm)\left[\frac{\frac{\cos 2\theta}{a^2}-\frac{\cos 2\omega}{b^2}}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}}+\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\left\{\frac{u^2}{a^4}+\frac{v^4}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},\right]$$

$$\frac{1}{R}-\frac{1}{R_1}=(\mp)\frac{a^2-b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}.$$

III.

Sectionibus normalibus, quae curvaturam quod attinet maximum et minimum praebeant, nunc occupabimur.

Sectiones eae brevitatis causa principales nominentur.

Sectiones principales ad perpendiculum inter se directas esse theoria docet generalis.

Itaque duas sectiones normales, quas modo tractavimus, si ponimus paraboloidis hyperbolicae esse principales, theoria generali inter Ω et Ω_1 formulas obtinemus has:

$$\tfrac{1}{2}(\Omega+\Omega_1)=\frac{1}{a^2}-\frac{1}{b^2}-\frac{\frac{u^2}{a^6}-\frac{v^2}{b^6}}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}$$

vel, quod jam supra demonstravimus,

$$\tfrac{1}{2}(\Omega+\Omega_1)=-\frac{a^2-b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

$$\tfrac{1}{4}\Omega\Omega_1=-\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{-1};$$

unde:

$$\Omega=\frac{\left\{-\{a^2-b^2\mp 2cw\}\,(\pm)\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}$$

et

$$\Omega_1=\frac{\left\{-\{a^2-b^2\mp 2cw\}\,(\mp)\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

ubi signa (uncis inclusa) in se referenda sunt.

Ex aequatione

$$\tfrac{1}{4}\Omega\Omega_1=-\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{u^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{-1}$$

Ω et Ω_1 diversis semper signis praedita esse quum appareat, pro radiis inversis osculatoriis sectionum principalium e §. II obtinemus expressiones has:

$$\frac{1}{R}=(\pm)\frac{\tfrac{1}{2}\Omega}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R_1}=(\mp)\frac{\tfrac{1}{2}\Omega_1}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ubi signa in se invicem referenda et superiora aut inferiora sumenda sunt, prout Ω affirmativum aut negativum prodit.

Eadem determinatione signorum est:

$$\frac{1}{R}+\frac{1}{R_1}=(\pm)\frac{\tfrac{1}{2}(\Omega-\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\frac{1}{R}-\frac{1}{R_1}=(\pm)\frac{\frac{1}{2}(\Omega+\Omega_1)}{\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ergo, quum facillime perspiciatur eadem signorum determinatione esse

$$\tfrac{1}{2}(\Omega-\Omega_1)=(\pm)\frac{\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}},$$

substitutione facta pro quantitate $\frac{1}{R}+\frac{1}{R_1}$ obtinemus expressionem:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

pro quantitate vero $\frac{1}{R}-\frac{1}{R_1}$ hanc:

$$\frac{1}{R}-\frac{1}{R_1}=(\mp)\frac{a^2-b^2\mp 2cw}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

ubi, ut jam supra commemoravimus, signum superius aut inferius sumendum est, prout Ω positivum aut negativum prodit, vel, quod idem est, prout in expressione pro Ω data signum superius aut inferius valet.

Est porro

$$\frac{1}{RR_1}=\frac{\frac{1}{4}\Omega\,\Omega_1}{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}},$$

vel valore pro $\frac{1}{4}\Omega\Omega_1$ substituto:

$$\frac{1}{RR_1}=\frac{1}{a^2b^2c^2}\cdot\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{-2},$$

et, quum sit

$$R+R_1=\left\{\frac{1}{R}+\frac{1}{R_1}\right\}RR_1,$$

substituendo prodit:

$$R+R_1=\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{1}{2}}\cdot\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}.$$

In formulas superiores si inducimus perpendiculum P e §. II notum, facili calculo obtinemus:

$$\frac{1}{R}+\frac{1}{R_1}=\frac{c}{a^2b^2}\cdot\frac{P^2}{w^2}\cdot\sqrt{\frac{P^2}{w^2}\cdot\{a^2-b^2\mp 2cw\}^2+4a^2b^2},$$

$$R+R_1=\frac{1}{c}\cdot\frac{w^2}{P^2}\cdot\sqrt{\frac{P^2}{w^2}\cdot\{a^2-b^2\mp 2cw\}^2+4a^2b^2},$$

$$\text{v. abs.}\left(\frac{1}{R}-\frac{1}{R_1}\right)=\text{v. abs.}\ \frac{P^3}{w^3}\cdot\frac{a^2-b^2\mp 2cw}{a^2b^2:c},$$

$$\frac{1}{RR_1}=\frac{c^2}{a^2b^2}\cdot\frac{P^4}{w^4}$$

vel

$$RR_1=\frac{a^2b^2}{c^2}\cdot\frac{w^4}{P^4}$$

vel

$$a^2b^2:c^2=\frac{P^4}{w^4}\cdot RR_1$$

vel etiam:

$$P=(\pm)w\sqrt[4]{\frac{a^2b^2:c^2}{RR_1}},$$

ubi signum sumendum est ita, ut P sit positivum.

Jam vero in expressiones pro $\frac{1}{R}$ et $\frac{1}{R_1}$ supra datas si loco Ω et Ω_1 valores eorum inducimus neque negligimus, quae de signis diximus, quomodo sumenda ac referenda sint, facili consideratione pro radiis osculatoriis sectionum principalium in puncto (uvw) paraboloidis hyperbolicae expressiones invenimus memorabiles, ubi signa (uncis inclusa) superiora ad superiora, inferiora ad inferiora referenda sunt:

$$\frac{2}{R}=\frac{\left\{\begin{array}{l}\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\\ \quad(\mp)\{a^2-b^2\mp 2cw\}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}},$$

$$\frac{2}{R_1}=\frac{\left\{\begin{array}{l}\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}}\\ \quad(\pm)\{a^2-b^2\mp 2cw\}\end{array}\right\}}{a^2b^2c^2\left\{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}\right\}^{\frac{3}{2}}}.$$

Sectiones principales ad quantitates Ω et Ω_1 maximum alteram, alteram praebere minimum, theoria scimus generali.

Conditiones, quibus Ω et Ω_1 maxima et minima sunt, nunc constituemus.

Sunt:

$$\frac{d^2\Omega}{d\,\theta^2} = 2\sin\theta^2 \,.\, \frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}{\left\{\frac{v}{b^2}\cos\overline{\omega} \pm \frac{1}{c}\cos\omega\right\}^2} \,.\, (\Omega_1 - \Omega),$$

$$\frac{d^2\Omega_1}{d\theta_1{}^2} = 2\sin\theta_1{}^2 \,.\, \frac{\frac{u^2}{a^4}+\frac{v^2}{b^4}+\frac{1}{c^2}}{\left\{\frac{v}{b^2}\cos\overline{\omega}_1 \pm \frac{1}{c}\cos\omega_1\right\}^2} \,.\, (\Omega - \Omega_1),$$

unde ex principiis calculi differentialis concludimus:

si $\Omega - \Omega_1 > 0$ est, Ω maximum et Ω_1 minimnm est; sin $\Omega - \Omega_1 < 0$ est, Ω minimum et Ω_1 maximum est.

Porro etiam nobis sunt computandi anguli, quibus sectionum principalium situs determinatur.

Quem ad finem quantitates notae M, M', M'' antea explicentur necesse est.

Eae facile exsistunt:

$$M = \frac{2u\Omega(2+b^2\Omega)}{a^2b^2},$$

$$M' = \frac{2v\Omega(2-a^2\Omega)}{a^2b^2},$$

$$M'' = \mp\frac{2.(2-a^2\Omega)(2+b^2\Omega)}{a^2b^2c};$$

ergo:

$$M^2 + M'^2 + M''^2$$

$$= 4 \,.\, \frac{(2-a^2\Omega)^2.(2+b^2\Omega)^2}{a^4b^4} \,.\, \left\{\left(\frac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\frac{v\Omega}{2+b^2\Omega}\right)^2 + \frac{1}{c^2}\right\}.$$

Theoria igitur generali ad determinandos angulos θ, ω, $\overline{\omega}$, quibus sectionum principalium situs definitur, formulas obtinemus has:

$$\cos\theta = (\pm)\, \frac{\frac{u\Omega}{2-a^2\Omega}}{\left\{\left(\frac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\frac{v\Omega}{2+b^2\Omega}\right)^2 + \frac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\cos\omega = (\pm)\,\frac{\dfrac{v\Omega}{2+b^2\Omega}}{\left\{\left(\dfrac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\dfrac{v\Omega}{2+b^2\Omega}\right)^2 + \dfrac{1}{c^2}\right\}^{\frac{1}{2}}},$$

$$\cos\overline{\omega} = (\pm)\mp\,\frac{\dfrac{1}{c}}{\left\{\left(\dfrac{u\Omega}{2-a^2\Omega}\right)^2 + \left(\dfrac{v\Omega}{2+b^2\Omega}\right)^2 + \dfrac{1}{c^2}\right\}^{\frac{1}{2}}};$$

ubi signa superiora et inferiora in se invicem referenda sunt; sed utra sumantur, superiora an inferiora, nihil interest, quum iis una modo ac eadem linea recta per punctum (uvw) ducta definiatur. Formulis vero illis, quum expressio pro Ω data duplicis signi causa duos pro Ω eosque, ut elucet, reales reddat valores, pro angulis duos et reales quidem reddi valores, statim efficitur.

Coordinatae denique centrorum curvaturarum sectionum principalium determinantur formulis hisce:

$$X-u = 2.\frac{u\,b^2c^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}{\left\{a^2-b^2\mp 2cw\,(\mp)\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}\right\}},$$

$$Y-v = -2.\frac{v\,c^2a^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}{\left\{a^2-b^2\mp 2cw\,(\mp)\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}\right\}},$$

$$Z-w = \pm 2.\frac{a^2b^2c\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}{\left\{a^2-b^2\mp 2cw\,(\mp)\sqrt{\{a^2-b^2\mp 2cw\}^2+4a^2b^2c^2\left\{\dfrac{u^2}{a^4}+\dfrac{v^2}{b^4}+\dfrac{1}{c^2}\right\}}\right\}};$$

ubi signa (uncis inclusa) superiora uni sectioni principali, alteri inferiora valent.

In iis, quae abhinc sequuntur, etiam quaesituri sumus, quos valores anguli, quibus sectionum principalium situs determinatur, et radii sectionum earum osculatorii induant, si punctum (uvw) in intersectione plani cujusvis coordinatarum ac paraboloidis hyperbolicae positum est.

Eadem, quam saepe jam attulimus, causa ex aequationibus, quibus anguli illi in universum paraboloidi hyperbolicae determinantur,

(1) $$\frac{u}{a^2}\cos\theta - \frac{v}{b^2}\cos\omega \pm \frac{1}{c}\cos\overline{\omega} = 0,$$

(2) $$\pm\frac{1}{c}\left\{\frac{1}{a^2}+\frac{1}{b^2}\right\}\cos\theta\cos\omega - \frac{1}{b^2}\cdot\frac{u}{a^2}\cos\omega\cos\overline{\omega}$$
$$+\frac{1}{a^2}\cdot\frac{v}{b^2}\cos\overline{\omega}\cos\theta = 0,$$

(3) $$\cos\theta^2 + \cos\omega^2 + \cos\overline{\omega}^2 = 1,$$

angulos computabimus.

Quibus igitur in aequationibus primum si ponimus

$$u = 0,$$

aequationi (2) satisfaciunt valores

$$\cos\theta = 0$$

et

$$\pm(a^2+b^2)\cos\omega + cv\cos\overline{\omega} = 0.$$

Si est

$$\cos\theta = 0,$$

ex aequat. (1) et (3) facili calculo sequitur:

$$\cos\omega = (\pm)\frac{b^2}{\sqrt{b^4+c^2v^2}},$$

$$\cos\overline{\omega} = (\pm)\mp\frac{cv}{\sqrt{b^4+c^2v^2}};$$

ergo, quum sit

$$\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} = -\frac{b^2}{b^4+c^2v^2},$$

$$R = \frac{\{b^4+c^2v^2\}^{\frac{3}{2}}}{b^4c};$$

sin est

$$\pm(a^2+b^2)\cos\omega + cv\cos\overline{\omega} = 0$$

et valores, in quos hoc casu anguli θ, ω, $\overline{\omega}$ et radius curvaturae R abeunt, litteris θ_1, ω_1, $\overline{\omega}_1$; R_1 significantur, ex aequat. illa et (1) facili calculo efficitur:

$$\{(a^2+b^2)b^2 + c^2v^2\}\cos\omega_1 = 0,$$

unde

$$\cos\omega_1 = 0,$$

ideoque ex aeq. (1)

$$\cos\overline{\omega}_1 =$$

et ex aeq. (3)

$$\cos\theta_1 = (\pm)\,1;$$

ergo, quia

$$\frac{\cos\theta^2}{a^2} - \frac{\cos\omega^2}{b^2} = \frac{1}{a^2},$$

$$R_1 = a^2 . \frac{\{b^4 + c^2 v^2\}^{\frac{3}{2}}}{b^2 c}.$$

Deinde si ponimus

$$v = 0,$$

eodem facili calculo obtinemus:

$$\cos\theta = (\pm)\frac{a^2}{\sqrt{a^4 + c^2u^2}}, \quad \cos\omega = 0, \quad \cos\varpi = (\pm) \mp \frac{cu}{\sqrt{a^4 + c^2u^2}},$$

$$R = \frac{\{a^4 + c^2u^2\}^{\frac{3}{2}}}{a^4 c};$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = (\pm)\,1, \quad \cos\varpi_1 = 0,$$

$$R_1 = b^2 \frac{\{a^4 + c^2 u\}^{\frac{1}{2}}}{a^2 c};$$

tum si ponimus

$$w = 0,$$

longo atque molesto calculo formulas invenimus, quas supra in universum pro angulorum cosinibus radiisque osculatoriis dedimus.

Si vero ponimus

$$w = 0$$

unaque

$$u = 0, \qquad v = 0,$$

quod per aequationem paraboloidis hyperbolicae licet, obtinemus:

$$\cos\theta = (\pm)\,1, \quad \cos\omega = 0, \quad \cos\varpi = 0;$$

$$R = \frac{a^2}{c};$$

$$\cos\theta_1 = 0, \quad \cos\omega_1 = (\pm)\,1, \quad \cos\varpi_1 = 0;$$

$$R_1 = \frac{b^2}{c}.$$

Quum aequatio plani paraboloidem hyperbolicam in puncto ($u=0$, $v=0$, $w=0$) tangentis sit

$$z = 0,$$

ergo planum tangens cum plano (xy) ideoque normalis in puncto illo erecta cum z axe coincidat, paraboloidi hyperbolicae idem, quod ellipticae, facili argumentatione efficitur theorema:

Si punctum (uvw) in intersectione planorum (xy), (yz) et paraboloidis hyperbolicae positum est, altera linea, quae sita in plano paraboloidem hyperbolicam in puncto (uvw) tangente alteram sectionem principalem paraboloidis hyperbolicae in puncto eo determinat, ei semper axi coordinatarum parallela est, qui ad planum, in quo punctum jacet, est perpendicularis; altera vero linea, quae in eodem plano sita alteram sectionem principalem in eo puncto paraboloidis hyperbolicae determinat, ad axem illum directa est; sin punctum (uvw) in initium incidit coordinatarum systematis, sectiones, quae planis coordinatarum (xz) et (yz) paraboloidem hyperbolicam persecantibus efficiuntur, paraboloidis hyperbolicae sunt principales.

Errata.

pag. 14. lin. 26. lege $(2-a^2\Omega)^2(2-b^2\Omega)^2(2-c^2\Omega)^2$
pro $(2-a^2\Omega)(2-b^2\Omega)(2-c^2\Omega)$.
pag. 16. lin. extr. Dele punctum.
pag. 62. lin. 21. lege $(2-a^2\Omega)^2(2+b^2\Omega)^2(2+c^2\Omega)^2$
pro $(2-a^2\Omega)(2+b^2\Omega)(2+c^2\Omega)$.

VITA.

Axelius Theodorus Willert, fidei addictus evangelicae, Gryphisvaldiae a. d. XXIV m. Jan. a. XXXVII h. s. natus sum patre Carolo Gustavo, matre Luisa e gente Doltz, quam morte jam dudum mihi ereptam lugeo. Primis literarum elementis in ludo quodam publico urbis patriae imbutus adii gymnasium adhuc florens et maturitatis testimonium adeptus tempore paschali a. h. s. LVII in almam universitatem Gryphisvaldensem me contuli, ut mathematicis rebus ac naturalibus operam navarem. Per tria semestria disserentes audivi vv. ill. Baier, de Feilitzsch, Grunert, Muenter, vv. cell. Susemihl, Schmitz. Quo tempore exacto ad studia persequenda Berolinum petivi ibique interfui scholis vv. ill. Dove, Kummer, v. exp. Franz. Ubi tamen diu manere mihi non licebat, quod officio satisfacere militari debebam. Itaque vere a. h. s. LIX in urbem patriam reversus in velitum cohorte Pomerana per annum voluntarius fui. Simul per tria semestria audivi disserentes vv. ill. Grunert, Limpricht, Muenter, Schaefer.

Quibus omnibus viris optime de me meritis gratias quam maximas ago, ac potissimum praeceptori dilectissimo Grunert, qui summa liberalitate librorum suorum usum mihi concessit neque quaerenti unquam deerat, nec non vv. ill. lib. bar. de Feilitzsch, George, Muenter, quorum tanta in me fuit comitas, ut, postquam jam de alma universitate decessi, me scholas adire benevolentissime sinerent.

THESES.

I.

Ad naturae leges evolvendas calculum maximi esse momenti.

II.

Vi attractionis et repulsionis corpora utrum solida sint an fluida an expansibilia definiri non posse.

III.

Fundamentum systematis in historia naturali esse speciei notionem.

IV.

Nucleo cellulae plantarum magnam vim, quam recentiores quidam botanici ei tribuerunt, abjudicandam esse.

V.

Cognitionem quasi asymptotam esse veritatis.